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THE ELLIPTIC INTEGRAL IN ELECTROMAGNETIC THEORY[*]

BY

ALFRED GEORGE GREENHILL

The analytical complexity in the reduction of the elliptic integral in electromagnetism, as well as in most dynamical problems, arises in consequence of the appropriate integral of the third kind being of the circular form in LEGENDRE's terminology, and the elliptic parameter of JACOBI is then a fraction of the imaginary period.

The expression of the integral by the theta function would imply then a complex argument, and a table of the theta function would not be of complete utility unless made a triple entry table.

When required in a problem of electromagnetism it is the complete third elliptic integral which usually is sufficient for a solution, and this, as shown by LEGENDRE, can be expressed by elliptic integrals of the first and second kind, complete and incomplete; and for these the Table IX of LEGENDRE provides the material for a numerical evaluation.

As an important application we may cite the calculation of the mutual induction of two coaxial helices, employed in the ampère balance designed for weighing their electromagnetic attraction by the late Viriamu JONES and Professor AYRTON, and so arriving at an independent determination of the electrical units.

The present investigation was undertaken in the lifetime of Professor Viriamu JONES, with the object of exhibiting the result of his complete third elliptic integral in its simplest form, suitable for immediate numerical computation, and also to reconcile the conflicting notation of different writers on the subject by adopting MAXWELL's *Electricity and Magnetism* as the standard.

Incidentally the quadric transformation of LANDEN is required here so frequently that a digression has been made on the theory, and an elucidation submitted of its essential geometrical interpretation.

The references in the course of the work will be chiefly to —

MAXWELL, *Electricity and Magnetism.*

WEBSTER, *Electricity and Magnetism.*

GRAY, *Absolute Measurements in Electricity and Magnetism.*

CAYLEY, Proceedings of the London Mathematical Society, vol. 6.

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Viriamu JONES, Philosophical Magazine (1889), Philosophical Transactions (1891), Proceedings of the Royal Society (1897).

MINCHIN, Philosophical Magazine (1893), (1894).

BURNSIDE, Messenger of Mathematics, vol. 20 (1891).

Coleridge FARR, Proceedings of the Royal Society (1898).

NAGAOKA, Journal of the Tokyo College of Science, 16 (1903).

COFFIN, ROSA, and COHEN, *Electromagnetic Integrals*, Bulletin of the Bureau of Standards (1906).

Alexander RUSSELL, *Magnetic Field of the Helix*, Philosophical Magazine, April, 1907.

Notation and preliminary theory.

1. The elliptic integrals of the I, II, and III kind (abbreviated in the sequel to I. E. I.; II. E. I.; III. E. I.) are composed of differential elements of the form

$$(1)\qquad \frac{ds}{\sqrt{S}},\qquad (s-\sigma)\frac{ds}{\sqrt{S}},\qquad \frac{1}{s-\sigma}\frac{ds}{\sqrt{S}},$$

$$(2)\qquad S=4\cdot s-s_1\cdot s-s_2\cdot s-s_3,\qquad s_1>s_2>s_3;$$

but for analytical simplification it is convenient to normalise them to zero degree by an appropriate factor, so that the elements may be written

$$(3)\qquad \frac{\sqrt{(s_1-s_3)}ds}{\sqrt{S}},\qquad \frac{s-\sigma}{\sqrt{(s_1-s_3)}}\frac{ds}{\sqrt{S}},\qquad \frac{\frac{1}{2}\sqrt{\Sigma}}{s-\sigma}\frac{ds}{\sqrt{S}},$$

$$(4)\qquad \Sigma=4\cdot\sigma-s_1\cdot\sigma-s_2\cdot\sigma-s_3;$$

and in the circular form of the III. E. I. the expression Σ is negative and the normalising factor is changed to $\frac{1}{2}\sqrt{(-\Sigma)}$.

The same normalising is required with the elementary circular or hyperbolic integral which arises when S is of the second or first degree, in consequence of s_1, or s_1 and s_2 being made infinite, so that we take

$$(5)\qquad S=4\cdot s-s_2\cdot s-s_3,\qquad \text{or}\qquad 4\cdot s-s_3;$$

and the integral corresponding to the III. E. I. becomes

$$(6)\qquad \int\frac{\sqrt{(s_2-\sigma\cdot\sigma-s_3)}}{s-\sigma}\frac{ds}{\sqrt{(4\cdot s-s_2\cdot s-s_3)}}=\tan^{-1}\sqrt{\left(\frac{s_2-\sigma}{\sigma-s_3}\cdot\frac{s-s_3}{s-s_2}\right)},$$

$$(7)\qquad \int\frac{\sqrt{(s_3-\sigma)}}{s-\sigma}\frac{ds}{\sqrt{(4\cdot s-s_3)}}=\sin^{-1}\sqrt{\frac{s-s_3}{s-\sigma}}=\tan^{-1}\sqrt{\frac{s-s_3}{s_3-\sigma}},$$

with corresponding hyperbolic-logarithmic forms.

The standard III. E. I. adopted in the sequel is written

$$(8) \qquad \int \frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s-\sigma}\,\frac{ds}{\sqrt{S}},$$

in which the sign of Σ is to be changed in the infrequent case which arises when the hyperbolic-logarithmic III. E. I. is to be employed; and it is preferable to keep to this form of the integral of an algebraic function, and to settle the sequence of magnitude of

$$\sigma,\ s,\ s_1,\ s_2,\ s_3,$$

before proceeding to a reduction in the notation of WEIERSTRASS, JACOBI, and LEGENDRE to their standard form, if requisite for comparison.

In an electromagnetic application it is the complete III. E. I. which gives the solution, and then the limits of integration are ∞, s_1, s_2, s_3, $-\infty$; and now the great theorem discovered by LEGENDRE (*Fonctions elliptiques*, I, chap. 23) enables us to express the result by incomplete integrals of the I and II kind, the I. E. I. and II. E. I., on which a digression is made at the outset.

2. The jacobian quarter periods, K and K', are defined by the complete integrals

$$(1) \qquad K = \int_{s_1,\, s_3}^{\infty,\, s_2} \frac{\sqrt{(s_1-s_3)}\,ds}{\sqrt{S}}, \qquad K' = \int_{s_2,\, -\infty}^{s_1,\, s_3} \frac{\sqrt{(s_1-s_3)}\,d\sigma}{\sqrt{(-\Sigma)}};$$

and the incomplete integrals may be written

$$(2) \qquad hK = \int_{s,\, s_3}^{\infty,\, s} \frac{\sqrt{(s_1-s_3)}\,ds}{\sqrt{S}}, \qquad fK' = \int_{\sigma,\, -\infty}^{s_1,\, \sigma} \frac{\sqrt{(s_1-s_3)}\,d\sigma}{\sqrt{(-\Sigma)}},$$

where h and f are real proper fractions defined by the ratios

$$(3) \qquad h = \int_{s,\, s_3}^{\infty,\, s} \div \int_{s_1,\, s_3}^{\infty,\, s_2} \frac{ds}{\sqrt{S}}, \qquad f = \int_{\sigma,\, -\infty}^{s_1,\, \sigma} \div \int_{s_2,\, -\infty}^{s_1,\, s_3} \frac{d\sigma}{\sqrt{(-\Sigma)}}.$$

In the jacobian notation

$$(4) \qquad k^2 = \frac{s_2-s_3}{s_1-s_3}, \qquad k'^2 = \frac{s_1-s_2}{s_1-s_3},$$

and when

$$\infty > s > s_1, \qquad \text{or} \qquad s_2 > s > s_3,$$

$$(5) \qquad \operatorname{sn}^2 hK = \frac{s_1-s_3}{s-s_3}, \qquad \text{or} \qquad \frac{s-s_3}{s_2-s_3},$$

$$(6) \qquad \operatorname{cn}^2 hK = \frac{s-s_1}{s-s_3}, \qquad \text{or} \qquad \frac{s_2-s}{s_2-s_3},$$

$$(7)\qquad \mathrm{dn}^2 hK = \frac{s - s_2}{s - s_3}, \quad\text{or}\quad \frac{s_1 - s}{s_1 - s_3};$$

while with

$$s_1 > \sigma > s_2, \quad\text{or}\quad s_3 > \sigma > -\infty,$$

$$(8)\qquad \mathrm{sn}^2 fK' = \frac{s_1 - \sigma}{s_1 - s_2}, \quad\text{or}\quad \frac{s_1 - s_3}{s_1 - \sigma},$$

$$(9)\qquad \mathrm{cn}^2 fK' = \frac{\sigma - s_2}{s_1 - s_2}, \quad\text{or}\quad \frac{s_3 - \sigma}{s_1 - \sigma},$$

$$(10)\qquad \mathrm{dn}^2 fK' = \frac{\sigma - s_3}{s_1 - s_3}, \quad\text{or}\quad \frac{s_2 - \sigma}{s_1 - \sigma}.$$

It is the advantage of the use of the fraction h or f of the period K or K that the corresponding modulus k or k' is indicated thereby and need not be written down.

In the notation given already in equations (2) the incomplete I. E. I. can be written

$$(11)\qquad hK = \int_{s_1,\, s_3}^{\infty,\, s} \frac{\sqrt{(s_1 - s_3)}\, ds}{\sqrt{S}} = \mathrm{sn}^{-1}\sqrt{\frac{s_1 - s_3}{s - s_3}}, \qquad \mathrm{sn}^{-1}\sqrt{\frac{s - s_3}{s_2 - s_3}};$$

$$(12)\quad (1 - h)K = \int_{s_1,\, s}^{s,\, s_2} \frac{\sqrt{(s_1 - s_3)}\, ds}{\sqrt{S}} = \mathrm{sn}^{-1}\sqrt{\frac{s - s_1}{s - s_2}}, \qquad \mathrm{dn}^{-1}\sqrt{\frac{s_1 - s_2}{s_1 - s}};$$

and for the complementary modulus k',

$$(13)\qquad fK' = \int_{\sigma,\, -\infty}^{s_1,\, \sigma} \frac{\sqrt{(s_1 - s_3)}\, d\sigma}{\sqrt{(-\Sigma)}} = \mathrm{sn}^{-1}\sqrt{\frac{s_1 - \sigma}{s_1 - s_2}}, \qquad \mathrm{sn}^{-1}\sqrt{\frac{s_1 - s_3}{s_1 - \sigma}};$$

$$(14)\quad (1 - f)K' = \int_{s_2,\, \sigma}^{\sigma,\, s_3} \frac{\sqrt{(s_1 - s_3)}\, d\sigma}{\sqrt{(-\Sigma)}} = \mathrm{sn}^{-1}\sqrt{\frac{\sigma - s_2}{\sigma - s_3}}, \qquad \mathrm{sn}^{-1}\sqrt{\frac{s_3 - \sigma}{s_2 - \sigma}}.$$

3. Besides the first elliptic integral (I. E. I.), given already in § 2 (1), (2), (11), (12), (13), (14), we shall require the second elliptic integral (II. E. I.) complete and incomplete.

The jacobian zeta function which expresses the incomplete II. E. I. may be efined by the standard integral

$$(1)\qquad \int_{s_3}^{s} \frac{s_1 - s}{\sqrt{(s_1 - s_3)}} \frac{ds}{\sqrt{S}} = \int_0^h \mathrm{dn}^2 hK \cdot K dh = E \,\mathrm{am}\, hK = hH + \mathrm{zn}\, hK,$$

or

$$(2)\quad \int_{s_2}^{\sigma} \frac{\sigma - s_3}{\sqrt{(s_1 - s_3)}} \frac{d\sigma}{\sqrt{(-\Sigma)}} = \int_0^f \mathrm{dn}^2 fK' \cdot K' df = E \,\mathrm{am}\, fK' = fH' + \mathrm{zn}\, fK';$$

where zn u in Glaisher's notation denotes the jacobian function Zu; also H and H' denote the complete II. E. I. to moduli k and k', defined by

$$(3) \qquad H = \int_0^1 \mathrm{dn}^2\, hK \cdot K dh, \qquad H' = \int_0^1 \mathrm{dn}^2 fK' \cdot K' df,$$

corresponding to K and K', the complete I. E. I.

The letter H is used instead of the usual E, as E will be required for the quadric transformation.

With Glaisher's notation

$$(4) \qquad \mathrm{zs}\, u = \mathrm{zn}\, u + \frac{d}{du} \log \mathrm{sn}\, u,$$

the incomplete II. E. I. in the regions

$$\infty > s > s_1 > s_2 > s > s_3,$$

is given by

$$(5) \quad \int_{s_1}^{s} \frac{s - s_1}{\sqrt{(s_1 - s_3)}} \frac{ds}{\sqrt{S}} = \int_{s}^{s_2} \frac{s_2 - s}{s - s_3} \frac{\sqrt{(s_1 - s_3)}\, ds}{\sqrt{S}} = -(1 - h) H + \mathrm{zs}\, hK;$$

$$(6) \quad \int \frac{s - s_2}{\sqrt{(s_1 - s_3)}} \frac{ds}{\sqrt{S}} = k^2 \int \frac{s_1 - s}{s - s_3} \frac{\sqrt{(s_1 - s_3)} ds}{\sqrt{S}} = -(1 - h)(H - k'^2 K) + \mathrm{zs}\, hK;$$

$$(7) \quad \int \frac{s - s_3}{\sqrt{(s_1 - s_3)}} \frac{ds}{\sqrt{S}} = \int \frac{s_2 - s_3}{s - s_3} \frac{\sqrt{(s_1 - s_3)}\, ds}{\sqrt{S}} = (1 - h)(K - H) + \mathrm{zs}\, hK;$$

the integrals being infinite at the upper limit $s = \infty$, or the lower limit $s = s_3$, where $h = 0$ and zs hK is infinite; and so also

$$(8) \quad \int_{s,\, s_1}^{\infty,\, s} \frac{s - s_2}{s - s_3} \frac{\sqrt{(s_1 - s_3)}\, ds}{\sqrt{S}} = \int_{s_3,\, s}^{s,\, s_2} \frac{s_1 - s}{\sqrt{(s_1 - s_3)}} \frac{ds}{\sqrt{S}}$$
$$= hH + \mathrm{zn}\, hK, \ (1 - h) H - \mathrm{zn}\, hK;$$

$$(9) \quad \int \frac{s - s_1}{s - s_3} \frac{\sqrt{(s_1 - s_3)}\, ds}{\sqrt{S}} = \int \frac{s_2 - s}{\sqrt{(s_1 - s_3)}} \frac{ds}{\sqrt{S}}$$
$$= h(H - k'^2 K) + \mathrm{zn}\, hK, \ (1 - h)(H - k'^2 K) - \mathrm{zn}\, hK;$$

$$(10) \quad \int \frac{s_2 - s_3}{s - s_3} \frac{\sqrt{(s_1 - s_3)}\, ds}{\sqrt{S}} = \int \frac{s - s_3}{\sqrt{(s_1 - s_3)}} \frac{ds}{\sqrt{S}}$$
$$= h(K - H) - \mathrm{zn}\, hK, \ (1 - h)(K - H) + \mathrm{zn}\, hK.$$

Similarly, in the regions

$$s_1 > \sigma > s_2 > s_3 > \sigma > -\infty,$$

$$(11)\quad \int_{\sigma,\,s_2}^{s_1,\,\sigma} \frac{s_1-\sigma}{\sqrt{(s_1-s_3)}}\,\frac{d\sigma}{\sqrt{(-\Sigma)}} = \int_{-\infty,\,\sigma}^{\sigma,\,s_3} \frac{s_1-s_2}{s_1-\sigma}\,\frac{\sqrt{(s_1-s_3)}\,d\sigma}{\sqrt{(-\Sigma)}}$$
$$= f(K'-H') - \mathrm{zn}\, fK',\ (1-f)(K'-H') + \mathrm{zn}\, fK';$$

$$(12)\quad \int \frac{\sigma-s_2}{\sqrt{(s_1-s_3)}}\,\frac{d\sigma}{\sqrt{(-\Sigma)}} = k'^2 \int \frac{s_3-\sigma}{s_1-\sigma}\,\frac{\sqrt{(s_1-s_3)}\,d\sigma}{\sqrt{(-\Sigma)}}$$
$$= f(H'-k^2K') + \mathrm{zn}\, fK',\ (1-f)(H'-k^2K') - \mathrm{zn}\, fK';$$

$$(13)\quad \int \frac{\sigma-s_3}{\sqrt{(s_1-s_3)}}\,\frac{d\sigma}{\sqrt{(-\Sigma)}} = \int \frac{s_2-\sigma}{s_1-\sigma}\,\frac{\sqrt{(s_1-s_3)}\,d\sigma}{\sqrt{(-\Sigma)}}$$
$$= fH' + \mathrm{zn}\, fK',\ (1-f)H' - \mathrm{zn}\, fK';$$

$$(14)\quad \int_{s_2}^{\sigma} \frac{s_1-s_2}{s_1-\sigma}\,\frac{\sqrt{(s_1-s_3)}\,d\sigma}{\sqrt{(-\Sigma)}} = \int_{\sigma}^{s_3} \frac{s_1-\sigma}{\sqrt{(s_1-s_3)}}\,\frac{d\sigma}{\sqrt{(-\Sigma)}}$$
$$= (1-f)(K'-H') + \mathrm{zs}\, fK';$$

$$(15)\quad k'^2 \int \frac{s_3-\sigma}{s_1-\sigma}\,\frac{\sqrt{(s_1-s_3)}\,d\sigma}{\sqrt{(-\Sigma)}} = \int \frac{s_2-\sigma}{\sqrt{(s_1-s_3)}}\,\frac{d\sigma}{\sqrt{(-\Sigma)}}$$
$$= -(1-f)(H'-k^2K') + \mathrm{zs}\, fK',$$

$$(16)\quad \int \frac{s_2-\sigma}{s_1-\sigma}\,\frac{\sqrt{(s_1-s_3)}\,d\sigma}{\sqrt{(-\Sigma)}} = \int \frac{s_3-\sigma}{\sqrt{(s_1-s_3)}}\,\frac{d\sigma}{\sqrt{(-\Sigma)}}$$
$$= -(1-f)H' + \mathrm{zs}\, fK';$$

these integrals (14), (15), (16) being infinite when the upper limit $\sigma = s_1$, or the lower limit $\sigma = -\infty$, where $f = 0$ and $\mathrm{zs}\, fK'$ is infinite.

Putting $h = 1$ or $f = 1$ in any of these forms will give the corresponding complete II. E. I., noticing that $\mathrm{zn}\, K'$ and $\mathrm{zs}\, K'$ are zero.

4. The incomplete III. E. I. requires the theta and eta functions of JACOBI, and BURKHARDT has given in his *Elliptische Functionen*, § 126, a method of approximating rapidly to the numerical value for given elliptic argument.

But in the circular III. E. I. this argument will be complex of the form $hK + fK'i$, where h and f are real fractions, so that great care will be required with BURKHARDT's method in separating the real and imaginary parts of the theta function.

When f, however, is a rational fraction,

$$(1)\qquad f = \tfrac{1}{2},\ \tfrac{1}{3},\ \tfrac{1}{4},\ \tfrac{1}{5},\ \cdots,$$

a quotient of theta functions, such as

$$(2)\qquad \frac{\Theta(hK+fK'i)}{\Theta(hK-fK'i)},\qquad \frac{\Theta 0\,\Theta(hK+fK'i)}{\Theta hK\,\Theta fK'i},$$

can be expressed algebraically by elliptic functions of hK; a collection of the simplest results will be found in the Philosophical Transactions (1904), under the title, *The third elliptic integral and the ellipsotomic problem.*

5. Make a start now with the complete circular III. E. I.

$$(1)\qquad A,\ B=\int_{s_1,\,s_3}^{\infty,\,s_2}\frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s\sim\sigma}\,\frac{ds}{\sqrt{S}},$$

in which the quantities range in the order of magnitude

$$(2)\qquad \infty>s>s_1>\sigma>s_2>s>s_3>-\infty$$

and this integral (1) is by LEGENDRE's theorem to be expressed by incomplete integrals, I. E. I. and II. E. I.

Provided with the notation above we shall find in the sequel

$$(3)\qquad A=\int_{s_1}^{\infty}\frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s-\sigma}\,\frac{ds}{\sqrt{S}}=\tfrac{1}{2}\pi(1-f)-K\,\mathrm{zn}\,fK',$$

$$(4)\qquad B=\int_{s_3}^{s_2}\frac{\frac{1}{2}\sqrt{(-\Sigma)}}{\sigma-s}\,\frac{ds}{\sqrt{S}}=\tfrac{1}{2}\pi f+K\,\mathrm{zn}\,fK',$$

$$(5)\qquad A+B=\tfrac{1}{2}\pi.$$

To calculate the numerical value we have to determine the co-modular angle θ and the amplitude angle of ϕ from the relations (2) and (8), § 2,

$$(6)\qquad \sin^2\theta=k'^2=\frac{s_1-s_2}{s_1-s_3},\qquad \sin^2\phi=\frac{s_1-\sigma}{s_1-s_2},$$

and then from Table IX of LEGENDRE

$$(7)\qquad f=F\phi\div F(\tfrac{1}{2}\pi),$$

$$(8)\qquad \mathrm{zn}\,fK'=E\phi-fE(\tfrac{1}{2}\pi),$$

$$(9)\qquad K=F(\tfrac{1}{2}\pi,k).$$

The result in the Weierstrassian notation is given in SCHWARZ's *Formeln*, §§ 59, 60.

The relation in (5) is the equivalent of MINCHIN's equation (21), p. 212, Philosophical Magazine, February, 1894; it is proved immediately by the substitution

$$(10)\qquad \left(s - s_3,\ \frac{s_1 - s_3 \cdot s_2 - s_3}{s - s_3}\right)$$

in A, (1), which leaves $ds/\sqrt{S}$ unaltered except for sign, but changes the limits from ∞ and s_1 to s_3 and s_2, and thus makes

$$(11)\qquad A = \int_{s_3}^{s_2} \frac{\frac{1}{2}\sqrt{(-\Sigma)}}{\dfrac{s_1 - s_3 \cdot s_2 - s_3}{s - s_3} - \sigma + s_3} \frac{ds}{\sqrt{S}},$$

$$(12)\quad A + B = \int \frac{\frac{1}{2}\sqrt{(-\Sigma)}\,[(s_1 - s_3)(s_2 - s_3) - (s - s_3)^2]}{[(s_1 - s_3 \cdot s_2 - s_3) - (s - s_3 \cdot \sigma - s_3)](\sigma - s)} \frac{ds}{\sqrt{S}}$$

$$= \cos^{-1}\sqrt{\frac{\sigma - s_3 \cdot s_1 - s \cdot s_2 - s}{[(s_1 - s_3)(s_2 - s_3) - (s - s_3)(\sigma - s_3)](\sigma - s)}}$$

$$= \sin^{-1}\sqrt{\frac{s_1 - \sigma \cdot \sigma - s_2 \cdot s - s_3}{[(s_1 - s_3 \cdot s_2 - s_3) - (s - s_3)(\sigma - s_3)](\sigma - s)}}$$

$$= \tfrac{1}{2}\sin^{-1} \frac{\frac{1}{2}\sqrt{(-\Sigma)}\sqrt{S}}{[(s_1 - s_3 \cdot s_2 - s_3) - (s - s_3)(\sigma - s_3)](\sigma - s)}\Bigg]_{s_3}^{s_2} = \tfrac{1}{2}\pi.$$

6. In the arrangement

$$(1)\qquad \infty > s > s_1 > s_2 > s > s_3 > \sigma > -\infty,$$

there are two remaining forms of the complete circular III. E. I., namely

$$(2)\qquad C,\ D = \int_{s_1,\,s_3}^{\infty,\,s_1} \frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s - \sigma} \frac{ds}{\sqrt{S}};$$

and we shall find

$$(3)\qquad C = \int_{s_1}^{\infty} \frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s - \sigma} \frac{ds}{\sqrt{S}} = K\,\mathrm{zs}\,fK' - \tfrac{1}{2}\pi(1 - f),$$

$$(4)\qquad D = \int_{s_3}^{s_2} \frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s - \sigma} \frac{ds}{\sqrt{S}} = K\,\mathrm{zs}\,fK' + \tfrac{1}{2}\pi f,$$

where in Glaisher's notation

$$(5)\qquad \mathrm{zs}\,u = \mathrm{zn}\,u + \frac{\mathrm{cn}\,u\,\mathrm{dn}\,u}{\mathrm{sn}\,u} = \mathrm{zn}\,u + \frac{d}{du}\log \mathrm{sn}\,u = \frac{d}{du}\log Hu,$$

and also

$$(6)\qquad \mathrm{zc}\,u = \mathrm{zn}\,u + \frac{d}{du}\log \mathrm{cn}\,u = -\,\mathrm{zs}(K - u),$$

$$(7)\qquad \mathrm{zd}\,u = \mathrm{zn}\,u + \frac{d}{du}\log \mathrm{dn}\,u = -\,\mathrm{zn}(K - u),$$

Thus

$$(8) \qquad -C + D = \tfrac{1}{2}\pi,$$

which can be proved independently in the same manner as (12) § 5.

Hence we can write down the expression of the complete III. E. I. of the circular form, for any assigned region of s and σ, the result involving a zeta function to the complementary modulus; and this zeta function is an algebraical function when f is a rational fraction; the formula for the simplest fractions is given in the Philosophical Transactions (1904).

7. We have seen how A and B change place by the substitution

$$(1) \qquad \left(s - s_3,\ \frac{s_1 - s_3 \cdot s_2 - s_3}{s - s_3}\right),$$

and the same substitution interchanges C and D.

So also A and C or B and D change place by the substitution

$$(2) \qquad \left(s - s_1,\ \frac{s_1 - s_2 \cdot s_1 - s_3}{s - s_1}\right);$$

while the substitution

$$(3) \qquad \left(s - s_2,\ \frac{s_2 - s_1 \cdot s_2 - s_3}{s - s_2}\right)$$

interchanges A and D, or B and C.

8. Two more forms are required to finish the series of the complete III. E. I., when it is finite and the parameter is a fraction of the real period, namely,

$$(1) \qquad E = \int_{s_1}^{\infty} \frac{\frac{1}{2}\sqrt{\Sigma}}{s - \sigma}\,\frac{ds}{\sqrt{S}} = K\,\mathrm{zn}\,fK,$$

$$(2) \qquad F = \int_{s_3}^{s_2} \frac{\frac{1}{2}\sqrt{S}}{s - \sigma}\,\frac{d\sigma}{\sqrt{\Sigma}} = K\,\mathrm{zs}\,hK,$$

in the arrangement

$$(3) \qquad \infty > s > s_1 > s_2 > \sigma > s_3 > -\infty,$$

with

$$(4) \qquad h = \int_{s}^{\infty} \div \int_{s_1}^{\infty} \frac{ds}{\sqrt{S}},$$

$$(5) \qquad f = \int_{s_3}^{s} \div \int_{s_3}^{s_2} \frac{d\sigma}{\sqrt{\Sigma}}.$$

(Legendre, *Fonctions elliptiques*, chap. 23; Cayley, *Elliptic Functions*, §§ 178–186.)

9. The proof of his theorem that the complete III. E. I., circular or logarithmic, can be expressed by the I. and II. E. I. is given by Legendre in *Fonctions elliptiques*, chapters 23 and 24, and is reproduced in Cayley's *Elliptic*

Functions, chap. 5. We can abbreviate the demonstration considerably, and avoid also all introduction of an imaginary in the discussion of the circular form expressed by A, B C, and D by means of the simple lemma

$$(1)\qquad \sqrt{S}\frac{d}{ds}\frac{\frac{1}{2}\sqrt{S}}{s\sim\sigma}-\sqrt{(-\Sigma)}\frac{d}{d\sigma}\frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s\sim\sigma}=s\sim\sigma,$$

proved immediately by performing the differentiation.

Integrating with respect to the elliptic differential elements $ds/\sqrt{S}$ and $d\sigma/\sqrt{(-\Sigma)}$ beween the limits s_3, s_2 of s, and σ, s_1 of σ,

$$\int\int\frac{d\sigma}{\sqrt{(-\Sigma)}}\frac{ds}{\sqrt{S}}\sqrt{S}\frac{d}{ds}\frac{\frac{1}{2}\sqrt{S}}{\sigma-s}-\int\int\frac{ds}{\sqrt{S}}\frac{d\sigma}{\sqrt{(-\Sigma)}}\sqrt{(-\Sigma)}\frac{d}{d\sigma}\frac{\frac{1}{2}\sqrt{(-\Sigma)}}{-s}$$

$$(2)\qquad =\int_\sigma^{s_1}\left(\frac{\frac{1}{2}\sqrt{S}}{\sigma-s}\right)_{s_3}^{s_2}\frac{d\sigma}{\sqrt{(-\Sigma)}}(\text{or } 0)+\int_{s_3}^{s_2}\frac{\frac{1}{2}\sqrt{(-\Sigma)}}{\sigma-s}\frac{ds}{\sqrt{S}}(\text{or } B)$$

$$=\int\int(\sigma-s)\frac{ds}{\sqrt{S}}\frac{d\sigma}{\sqrt{(-\Sigma)}},$$

in which the variables are separated, so that

$$B=\int_\sigma^{s_1}\int_{s_3}^{s_2}(\sigma-s_3)\frac{ds}{\sqrt{S}}\frac{d\sigma}{\sqrt{(-\Sigma)}}-\int\int(s-s_3)\frac{ds}{\sqrt{S}}\frac{d\sigma}{\sqrt{(-\Sigma)}}$$

$$(3)\qquad =K\int_\sigma^{s_1}\frac{\sigma-s_3}{\sqrt{(s_1-s_3)}}\frac{d\sigma}{\sqrt{(-\Sigma)}}-fK'\int_{s_3}^{s_2}\frac{s-s_3}{\sqrt{(s_1-s_3)}}\frac{ds}{\sqrt{S}}$$

$$=K(fH'+\operatorname{zn}fK')-fK'(K-H)$$

$$=\tfrac{1}{2}\pi f+K\operatorname{zn}fK',$$

by reason of LEGENDRE's relation

$$(4)\qquad \int_{s_2}^{s_1}\int_{s_3}^{s_2}(\sigma-s)\frac{ds}{\sqrt{S}}\frac{d\sigma}{\sqrt{(-\Sigma)}}=KH'+K'H-KK'=\tfrac{1}{2}\pi,$$

derivable from the relation in (5) § 5, and utilising theorems (13) and (10), § 3.

Taking the limits of s as s_3 and s gives a more general theorem

$$\int_\sigma^{s_1}\frac{\frac{1}{2}\sqrt{S}}{\sigma-s}\frac{d\sigma}{\sqrt{(-\Sigma)}}+\int_{s_3}^{s}\frac{\frac{1}{2}\sqrt{(-\Sigma)}}{\sigma-s}\frac{ds}{\sqrt{S}}$$

$$(5)\qquad =hK\int_\sigma^{s_1}\frac{\sigma-s_3}{\sqrt{(s_1-s_3)}}\frac{d\sigma}{\sqrt{(-\Sigma)}}-fK'\int_{s_3}^{s}\frac{s-s_3}{\sqrt{(s_1-s_3)}}\frac{ds}{\sqrt{S}}$$

$$=hK(fH'+\operatorname{zn}fK')-fK'\left[h(K-H)-\operatorname{zn}hK\right]$$

$$=\tfrac{1}{2}\pi hf+hK\operatorname{zn}fK'-fK'\operatorname{zn}hK,$$

connecting incomplete III. E.I's., as in LEGENDRE's equation (i'), p. 133.

Putting $h = 1$ gives B as before; and putting $f = 1$ leads to a new theorem

$$(6)\qquad \int_{s_2}^{s_1} \frac{\frac{1}{2}\sqrt{S}}{\sigma - s}\frac{d\sigma}{\sqrt{(-\Sigma)}} = \tfrac{1}{2}\pi h - K' \,\mathrm{zn}\, hK.$$

10. When A is deduced in this manner we must keep clear of $s = \infty$ by writing the lemma (1), § 9, in the form

$$(1)\quad \sqrt{(-\Sigma)}\frac{d}{d\sigma}\frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s-\sigma} - \sqrt{S}\frac{d}{ds}\left(\frac{\frac{1}{2}\sqrt{S}}{s-\sigma} - \frac{\frac{1}{2}\sqrt{S}}{s-s_3}\right) = \sigma - s_3 - \frac{s_1 - s_3 \cdot s_2 - s_3}{s - s_3},$$

in which at

$$(2)\qquad s = \infty, \qquad \text{limit}\left(\frac{\frac{1}{2}\sqrt{S}}{s-\sigma} - \frac{\frac{1}{2}\sqrt{S}}{s-s_3}\right) = 0.$$

Integrating between the limits s, ∞ of s, and s_2, σ of σ, we have

$$(3)\quad \int_s^\infty \frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s-\sigma}\frac{ds}{\sqrt{S}} + \int_{s_2}^{\sigma}\left(\frac{\frac{1}{2}\sqrt{S}}{s-\sigma} - \frac{\frac{1}{2}\sqrt{S}}{s-s_3}\right)\frac{d\sigma}{\sqrt{(-\Sigma)}}$$

$$= \int\!\!\int\left(\sigma - s_3 - \frac{s_1 - s_3 \cdot s_2 - s_3}{s - s_3}\right)\frac{ds}{\sqrt{S}}\frac{d\sigma}{\sqrt{(-\Sigma)}}$$

(in which the variables are separated)

$$= hK\int_{s_2}^{\sigma}\frac{\sigma - s_3}{\sqrt{(s_1 - s_3)}}\frac{d\sigma}{\sqrt{(-\Sigma)}} - (1-f)K'\int_s^\infty \frac{s_2 - s_3}{s - s_3}\frac{\sqrt{(s_1 - s_3)}ds}{\sqrt{S}}$$

$$= hK[(1-f)H' - \mathrm{zn}\, fK'] - (1-f)K'\,[h(K-H) - \mathrm{zn}\, hK]$$

$$= \tfrac{1}{2}\pi h(1-f) - hK\,\mathrm{zn}\, fK' + (1-f)K'\,\mathrm{zn}\, hK,$$

utilising the theorems in (13) and (5), § 3.

Putting $s = s_1$, $h = 1$, gives the result for A,

$$(4)\quad A = \int_{s_1}^\infty \frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s-\sigma}\frac{ds}{\sqrt{S}} = \int_{s_1}^\infty\int_{s_2}^{\sigma}\left(\sigma - s_3 - \frac{s_1 - s_3 \cdot s_2 - s_3}{s - s_3}\right)\frac{d\sigma}{\sqrt{(-\Sigma)}}\frac{ds}{\sqrt{S}},$$

and this by the substitution (10), § 5, is equivalent to

$$(5)\qquad A = \int_{s_2}^{\sigma}\int_{s_3}^{s_2}(\sigma - s)\frac{ds}{\sqrt{S}}\frac{d\sigma}{\sqrt{(-\Sigma)}},$$

while from (2), § 9,

$$(6)\qquad B = \int_{\sigma}^{s_1}\int_{s_3}^{s_2}(\sigma - s)\frac{ds}{\sqrt{S}}\frac{d\sigma}{\sqrt{(-\Sigma)}},$$

$$(7)\qquad A+B=\int_{s_2}^{s_1}\int_{s_3}^{s_2}(\sigma-s)\frac{ds}{\surd S}\,\frac{d\sigma}{\surd(-\Sigma)},$$

or from (12), § 5, and employing (12), (9), § 3,

$$(8)\qquad \tfrac{1}{2}\pi=\int\int[\sigma-s_2+s_2-s]\frac{ds}{\surd S}\,\frac{d\sigma}{\surd(-\Sigma)}$$

$$=K(H'-k^2K')+K'(H-k'^2K);$$

and thus LEGENDRE's relation is proved.

Putting $f=0$, $\sigma=s_1$ in (3) gives a new theorem,

$$(9)\ \int_{s_2}^{s_1}\frac{\tfrac{1}{2}\surd S}{s-\sigma}\,\frac{d\sigma}{\surd(-\Sigma)}=\tfrac{1}{2}\pi h+K'\operatorname{zn}hK+K'\sqrt{\frac{s-s_1\cdot s-s_2}{s-s_3\cdot s_1-s_3}}=\tfrac{1}{2}\pi h+K'\operatorname{zs}hK,$$

employing (5), (6), (7), § 2, and (4), § 3, or (5), § 6.

11. For C, integrate lemma (1), § 9, from s to ∞ and σ to s_3 in the form

$$(1)\ -\surd(-\Sigma)\frac{d}{d\sigma}\frac{\tfrac{1}{2}\surd(-\Sigma)}{s-\sigma}+\surd S\frac{d}{ds}\left(\frac{\tfrac{1}{2}\surd S}{s-\sigma}-\frac{\tfrac{1}{2}\surd S}{s-s_3}\right)=s_3-\sigma+\frac{s_1-s_3\cdot s_2-s_3}{s-s_3},$$

$$\int_s^\infty\frac{\tfrac{1}{2}\surd(-\Sigma)}{s-\sigma}\,\frac{ds}{\surd S}-\int_\sigma^{s_3}\left(\frac{\tfrac{1}{2}\surd S}{s-\sigma}-\frac{\tfrac{1}{2}\surd S}{s-s_3}\right)\frac{d\sigma}{\surd(-\Sigma)}$$

$$=\int\int\left(s_3-\sigma+\frac{s_1-s_3\cdot s_2-s_3}{s-s_3}\right)\frac{ds}{\surd S}\,\frac{d\sigma}{\surd(-\Sigma)}$$

$$(2)\qquad =hK\int_\sigma^{s_3}\frac{s_3-\sigma}{\surd(s_1-s_3)}\,\frac{d\sigma}{\surd(-\Sigma)}+(1-f)K'\int_s^\infty\frac{s_2-s_3}{s-s_3}\,\frac{\surd(s_1-s_3)\,ds}{\surd S}$$

$$=hK[\operatorname{zs}fK'-(1-f)H']+(1-f)K'[h(K-H)-\operatorname{zn}hK]$$

$$=-\tfrac{1}{2}\pi h(1-f)-(1-f)K'\operatorname{zn}hK+hK\operatorname{zs}fK',$$

employing (16), (10), § 3; from which C is deduced by putting $h=1$, $s=s_1$,

$$(3)\qquad C=-\tfrac{1}{2}\pi(1-f)+K\operatorname{zs}fK'.$$

In D we must keep clear of $\sigma=s_1$ by writing the lemma

$$(4)\ \ \surd(-\Sigma)\frac{d}{d\sigma}\left(\frac{\tfrac{1}{2}\surd(-\Sigma)}{s-\sigma}-\frac{\tfrac{1}{2}\surd(-\Sigma)}{s_1-\sigma}\right)-\surd S\frac{d}{ds}\cdot\frac{\tfrac{1}{2}\surd S}{s-\sigma}=s_1-s-\frac{s_1-s_2\cdot s_1-s_3}{s_1-\sigma}$$

and then integrate it between s_3 and s, $-\infty$ and σ, so that

$$\int_{s_3}^{s}\left(\frac{\tfrac{1}{2}\surd(-\Sigma)}{s-\sigma}-\frac{\tfrac{1}{2}\surd(-\Sigma)}{s_1-\sigma}\right)\frac{ds}{\surd S}-\int_{-\infty}^{\sigma}\frac{\tfrac{1}{2}\surd S}{s-\sigma}\,\frac{d\sigma}{\surd(-\Sigma)}$$

$$(5)\qquad = fK'\int\frac{s_1-s}{\sqrt{(s_1-s_3)}}\frac{ds}{\sqrt{S}}-hK\int\frac{s_1-s_2}{s_1-\sigma}\sqrt{(s_1-s_3)}\frac{d\sigma}{\sqrt{(-\Sigma)}}$$

$$= fK'(hH+\operatorname{zn}hK)-hK[f(K'-H')-\operatorname{zn}fK']$$

$$= \tfrac{1}{2}\pi hf+fK'\operatorname{zn}hK+hK\operatorname{zn}fK',$$

employing (8), (11), § 3.

Put $h=1$, $s=s_2$, and we obtain by means of (8), (9), (10), § 2,

$$(6)\quad D=\tfrac{1}{2}\pi f+K\operatorname{zn}fK'+K\sqrt{\frac{\sigma-s_2\cdot\sigma-s_3}{s_1-\sigma\cdot s_1-s_3}}=\tfrac{1}{2}\pi f+K\operatorname{zs}fK'.$$

12. To obtain E and F, write the lemma

$$(1)\quad \sqrt{\Sigma}\frac{d}{d\sigma}\frac{\frac{1}{2}\sqrt{\Sigma}}{s-\sigma}+\sqrt{S}\frac{d}{ds}\left(\frac{\frac{1}{2}\sqrt{S}}{s-\sigma}-\frac{\frac{1}{2}\sqrt{S}}{s-s_3}\right)=\frac{s_1-s_3\cdot s_2-s_3}{s-s_3}-\sigma+s_3,$$

and integrate between s, ∞ and s_3, σ; then

$$\int_s^\infty\frac{\frac{1}{2}\sqrt{\Sigma}}{s-\sigma}\frac{ds}{\sqrt{S}}-\int_{s_3}^\sigma\left(\frac{\frac{1}{2}\sqrt{S}}{s-\sigma}-\frac{\frac{1}{2}\sqrt{S}}{s-s_3}\right)\frac{d\sigma}{\sqrt{\Sigma}}$$

$$(2)\qquad = fK\int\frac{s_2-s_3}{s-s_3}\frac{\sqrt{(s_1-s_3)}\,ds}{\sqrt{S}}-hK\int\frac{\sigma-s_3}{\sqrt{(s_1-s_3)}}\frac{d\sigma}{\sqrt{\Sigma}}$$

$$= fK[h(K-H)-\operatorname{zn}hK]-hK[f(K-H)-\operatorname{zn}fK]$$

$$= hK\operatorname{zn}fK-fK\operatorname{zn}hK,$$

employing (10), § 3, and thus connecting E and F; for $h=1$, $s=s_1$, makes

$$(3)\qquad E=K\operatorname{zn}fK,$$

and $f=1$, $\sigma=s_2$, makes

$$(4)\qquad F=K\operatorname{zn}hK+K\sqrt{\frac{s-s_1\cdot s-s_2}{s-s_3\cdot s_1-s_3}}=K\operatorname{zs}hK.$$

13. With s_1 and s_3 conjugate imaginary, and putting

$$(1)\qquad s_1-s_2\cdot s_3-s_2=M^2,$$

$$(2)\qquad K=\int_{s_2}^\infty\frac{\sqrt{M}ds}{\sqrt{S}},\qquad hK=\int_{s_2}^s\frac{\sqrt{M}ds}{\sqrt{S}},$$

$$(3)\qquad K'=\int_{-\infty}^{s_2}\frac{\sqrt{M}d\sigma}{\sqrt{(-\Sigma)}},\qquad fK'=\int_\sigma^{s_2}\frac{\sqrt{M}d\sigma}{\sqrt{(-\Sigma)}},$$

and then

$$(4)\qquad s-s_2=M\frac{1-\operatorname{cn}2hK}{1+\operatorname{cn}2hK},\qquad \sqrt{S}=M^{\frac{3}{2}}\frac{4\operatorname{sn}2hK\operatorname{dn}2hK}{(1+\operatorname{cn}2hK)^2},$$

$$(5)\qquad s_2-\sigma=M\frac{1-\operatorname{cn}2fK}{1+\operatorname{cn}2fK},\qquad \surd(-\Sigma)=M^{\frac{3}{2}}\frac{4\operatorname{sn}2fK'\operatorname{dn}2fK'}{(1+\operatorname{cn}2fK')^2},$$

in which hK and fK' may be replaced occasionally by h and f in the sequel without confusion.

Employing a formula (LEGENDRE I, p. 257),

$$(6)\qquad \int\frac{1-\cos\phi}{1+\cos\phi}\frac{d\phi}{\Delta\phi}=\frac{2\sin\phi\Delta\phi}{1+\cos\phi}+F\phi-2E\phi,$$

verified immediately by differentiation, and integrating the lemma (1) § 9,

$$\int_\sigma^{s_2}\frac{\frac{1}{2}\surd S}{s-\sigma}\frac{d\sigma}{\surd(-\Sigma)}+\int_{s_2}^{s}\frac{\frac{1}{2}(\surd-\Sigma)}{s-\sigma}\frac{ds}{\surd S}=\int_\sigma^{s_2}\int_{s_2}^{s}(s-\sigma)\frac{ds}{\surd S}\frac{d\sigma}{\surd(-\Sigma)}$$

$$=fK'\int_{s_2}^{s}\frac{s-s_2}{M}\cdot\frac{\surd M ds}{\surd S}+hK\int_\sigma^{s_2}\frac{s_2-\sigma}{M}\frac{\surd M d\sigma}{\surd(-\Sigma)}$$

$$(7)\qquad =fK'\int_0^h\frac{1-\operatorname{cn}2h}{1+\operatorname{cn}2h}Kdh+hK\int_0^f\frac{1-\operatorname{cn}2f}{1+\operatorname{cn}2f}K'df$$

$$=fK'\left(\frac{\operatorname{sn}2h\operatorname{dn}2h}{1+\operatorname{cn}2h}+\tfrac{1}{2}hK-hH-\operatorname{zn}2h\right)$$

$$+hK\left(\frac{\operatorname{sn}2f\operatorname{dn}2f}{1+\operatorname{cn}2f}+\tfrac{1}{2}fK'-fH'-\operatorname{zn}2f\right).$$

But when we wish to obtain the complete III. E. I., by putting $s=\infty$, $h=1$, we make both sides infinite.

The infinite part must then be cut out by deducting from both sides

$$(8)\qquad \int_\sigma^{s_2}\frac{\frac{1}{2}\surd S}{s-s_2}\frac{d\sigma}{\surd(-\Sigma)}=2fK'\frac{\operatorname{dn}2h}{\operatorname{sn}2h},$$

and now

$$(9)\qquad \int_\sigma^{s}\left(\frac{\frac{1}{2}\surd S}{s-\sigma}-\frac{\frac{1}{2}\surd S}{s-s_2}\right)\frac{d\sigma}{\surd(-\Sigma)}+\int_{s_2}^{s}\frac{\frac{1}{2}\surd(-\Sigma)}{s-\sigma}\frac{ds}{\surd S}$$

$$=fK'\left(\frac{-\operatorname{sn}2h\operatorname{dn}2h}{1-\operatorname{cn}2h}+\tfrac{1}{2}hK-hH-\operatorname{zn}2h\right)$$

$$+hK\left(\frac{\operatorname{sn}2f\operatorname{dn}2f}{1+\operatorname{cn}2f}+\tfrac{1}{2}fK'-fH'-\operatorname{zn}2f\right).$$

Here, putting $s=\infty$, $h=1$, the first integral vanishes, and

$$(10)\qquad \int_{s_2}^{\infty}\frac{\frac{1}{2}\surd(-\Sigma)}{s-\sigma}\frac{ds}{\surd S}=fK'(\tfrac{1}{2}K-H)$$

$$+K\left(\frac{\operatorname{sn}2f\operatorname{dn}2f}{1+\operatorname{cn}2f}+\tfrac{1}{2}fK'-fH'-\operatorname{zn}2f\right)$$

$$= K\left(\frac{\operatorname{sn} 2f \operatorname{dn} 2f}{1 + \operatorname{cn} 2f} - \operatorname{zn} 2f\right) - \tfrac{1}{2}\pi f$$

$$= K\left(\frac{\operatorname{dn} 2f}{\operatorname{sn} 2f} - \operatorname{zs} 2f\right) - \tfrac{1}{2}\pi f.$$

Similarly with $f = 1$,

$$(11) \qquad \int_{-\infty}^{s_2} \frac{\frac{1}{2}\surd S}{s - \sigma} \frac{d\sigma}{\surd(-\Sigma)} = K'\left(\frac{\operatorname{dn} 2h}{\operatorname{sn} 2h} - \operatorname{zs} 2h\right) - \tfrac{1}{2}\pi h.$$

14. To compare B with LEGENDRE's standard form of the III. E. I.,

$$(1) \qquad \int_0^x \frac{1}{1 + n \sin^2\chi} \frac{d\chi}{\Delta\chi}, \qquad \Delta\chi = \surd(1 - k^2 \sin^2\chi),$$

denoted by $\Pi\,(n, k, \chi)$ or Π, when complete, put

$$(2) \quad s - s_3 = (s_2 - s_3)\sin^2\chi, \qquad s_2 - s = (s_2 - s_3)\cos^2\chi, \qquad s_1 - s = (s_1 - s_3)\Delta^2\chi,$$

$$k^2 = \frac{s_2 - s_3}{s_1 - s_3}, \qquad k'^2 = \frac{s_1 - s_2}{s_1 - s_3};$$

and then

$$(3) \qquad B = \sqrt{\frac{s_1 - \sigma \cdot \sigma - s_2}{s_1 - s_3 \cdot \sigma - s_3}} \int \frac{1}{1 + n \sin^2\chi} \frac{d\chi}{\Delta\chi} = \surd\alpha\,\Pi\,(n, k, \chi),$$

in which, with LEGENDRE's notation,

$$(4) \qquad n = -\frac{s_2 - s_3}{\sigma - s_3}, \qquad 1 + n = \frac{\sigma - s_2}{\sigma - s_3}, \qquad 1 + \frac{k^2}{n} = \frac{s_1 - \sigma}{s_1 - s_3},$$

$$(5) \qquad \alpha = (1 + n)\left(1 + \frac{k^2}{n}\right) = \frac{s_1 - \sigma \cdot \sigma - s_2}{s_1 - s_3 \cdot \sigma - s_3},$$

and LEGENDRE's Π requires the normalising factor $\surd\alpha$.

In the comparison of A, put

$$(6) \quad s - s_3 = \frac{s_1 - s_3}{\sin^2\chi}, \qquad s - s_2 = (s_1 - s_3)\frac{\Delta^2\chi}{\sin^2\chi}, \qquad s - s_1 = (s_1 - s_3)\frac{\cos^2\chi}{\sin^2\chi},$$

and then

$$(7) \quad A = \sqrt{\frac{s_1 - \sigma \cdot \sigma - s_2}{s_1 - s_3 \cdot \sigma - s_3}} \int \frac{-n' \sin^2\chi}{1 + n' \sin^2\chi} \frac{d\chi}{\Delta\chi} = \surd\alpha[\Pi\,(n', k, \chi) - F\chi]$$

in which

$$(8) \qquad n' = -\frac{\sigma - s_3}{s_1 - s_3}, \qquad 1 + n' = \frac{s_1 - \sigma}{s_1 - s_3}, \qquad 1 + \frac{k^2}{n'} = \frac{\sigma - s_2}{\sigma - s_3},$$

so that

$$(9) \qquad (1 + n')\left(1 + \frac{k^2}{n'}\right) = \frac{s_1 - \sigma \cdot \sigma - s_2}{s_1 - s_3 \cdot \sigma - s_3} = \alpha,$$

the same as before; also

(10) $$nn' = k^2, \qquad \Pi(n) + \Pi(n') = K + \frac{\frac{1}{2}\pi}{\sqrt{\alpha}}.$$

But in all the subsequent calculations it is best to keep clear of the standard forms of LEGENDRE, JACOBI, or WEIERSTRASS, and to work with the III. E. I. as the integral of an algebraical function of the standard form in (8) § 1.

For example we find, putting the differential element,

(11) $$\frac{ds}{\sqrt{S}} = du,$$

(12) $$\frac{d^2y}{du^2} = \sqrt{S}\frac{d}{ds}\left(\sqrt{S}\frac{dy}{ds}\right) = S\frac{d^2y}{ds^2} + \tfrac{1}{2}\frac{dS}{ds}\frac{dy}{ds};$$

and if we take

(13) $$\log y = \tfrac{1}{2}\int\frac{\sqrt{S}-\sqrt{\Sigma}}{s-\sigma}\,\frac{ds}{\sqrt{S}},$$

we obtain the Lamé differential equation of the second order

(14) $$\frac{1}{y}\frac{d^2y}{du^2} = 2s + \sigma - s_1 - s_2 - s_3,$$

(15) $$S\frac{d^2y}{ds^2} + \tfrac{1}{2}\frac{dS}{ds}\frac{dy}{ds} - (2s + \sigma - s_1 - s_2 - s_3)y = 0.$$

Then the function

(16) $$z = \frac{d}{du}(ye^{\lambda u})$$

leads to the Lamé differential equation of the second order

(17) $$\frac{1}{z}\frac{d^2z}{du^2} = 6s - 3\sigma - s_1 - s_2 - s_3 + 3\lambda^2,$$

provided that

(18) $$\lambda^3 - \lambda(3\sigma - s_1 - s_2 - s_3) - \sqrt{\Sigma} = 0,$$

the spherical pendulum relation.

Landen's Transformation.

15. It will be convenient at this stage to collect together all the formulas of Landen's transformation of the second order, required in the sequel, adopting the notation employed previously for the elliptic argument, hK denoting an argument to modulus k and fK' to modulus k', h and f denoting real fractions, as thereby the modulus is indicated in the notation and need not be written down.

Without this it will be difficult to reconcile the notation of different writers on this subject, each adopting a method of his own irrespective of others.

Following MAXWELL in his *Electricity and Magnetism* (E. and M.) § 702, we select the LANDEN transformation connecting the new modulus c and c' with k and k' by the relation

$$(1)\qquad c'=\frac{1-k}{1+k},\qquad c=\frac{2\sqrt{k}}{1+k},\qquad k=\frac{1-c'}{1+c'},$$

$$k'=\frac{2\sqrt{c'}}{1+c'},\,k'c=2\sqrt{(kc')},\qquad c>k,\qquad c'<k',$$

all different forms of the modular equation of the second order, connecting the modular angles γ and β shown in figures A and B, such that

$$(2)\qquad \begin{aligned}\cos\gamma=c'&=\frac{1-\sin\beta}{1+\sin\beta}=\tan^2(\tfrac{1}{4}\pi-\tfrac{1}{2}\beta)=\frac{BC}{AC},\\ k=\sin\beta&=\frac{1-\cos\gamma}{1-\cos\gamma}=\tan^2\tfrac{1}{2}\gamma=\frac{AC}{AC'}=\frac{OC}{OA}.\end{aligned}$$

Then the complete elliptic integral or quarter-period $F(\frac{1}{2}\pi,\, c)$ or $F(c)$ is denoted by F, and $F(c')$ by F', K and K' denoting $F(k)$ and $F(k')$; and

$$(3)\qquad F=(1+k)K,\qquad K=\tfrac{1}{2}(1+c')F,$$

$$(4)\qquad F'=\tfrac{1}{2}(1+k)K',\qquad K'=(1+c')F',$$

$$(5)\qquad \frac{F'}{F}=\tfrac{1}{2}\frac{K'}{K},\qquad \frac{K'}{K}=2\frac{F'}{F}.$$

16. The geometrical interpretation of LANDEN's transformation is seen in fig. B, where we may put

$$(1)\qquad \omega=\operatorname{am}hF,\qquad \omega'=\operatorname{am}(1-h)F,\qquad \chi=\operatorname{am}2hK,$$

$$(2)\qquad \theta=2\omega=\chi+\chi',\qquad 2\omega'=\pi-\chi+\chi',$$

so that

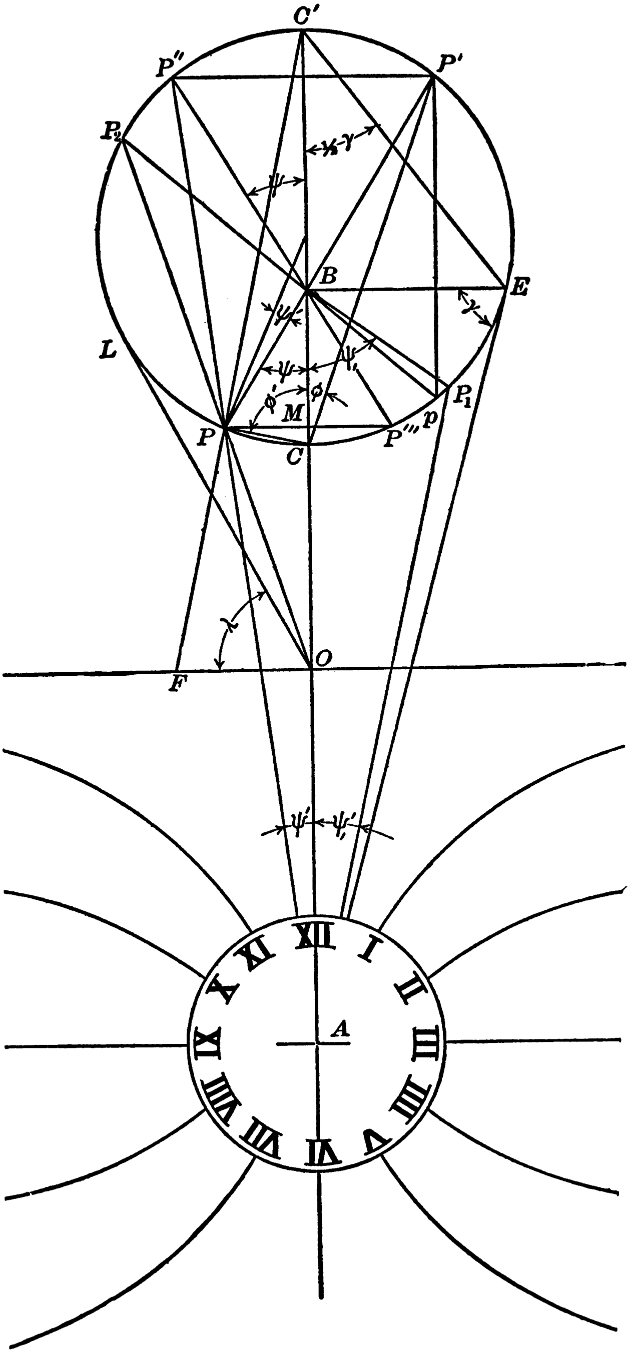
$$(3)\qquad \chi=\tfrac{1}{2}\pi+\omega-\omega',\qquad \operatorname{am}2hK=\tfrac{1}{2}\pi+\operatorname{am}hF-\operatorname{am}(1-h)F,$$

$$(4)\qquad \begin{aligned}\operatorname{sn}2hK&=\cos\left[\operatorname{am}hF-\operatorname{am}(1-h)F\right]\\ &=\operatorname{cn}hF\operatorname{cn}(1-h)F+\operatorname{sn}hF\operatorname{sn}(1-h)F=(1+c')\frac{\operatorname{sn}hF\operatorname{cn}hF}{\operatorname{dn}hK},\end{aligned}$$

LANDEN's first transformation; and putting $h=0$ leads to (3) § 15.

Thence also

$$(5)\qquad \begin{aligned}\operatorname{cn}2hK&=\frac{1-(1+c')\operatorname{sn}^2hF}{\operatorname{dn}hF}=\frac{\operatorname{dn}^2hF-c'}{(1-c')\operatorname{dn}hF}\\ &=\frac{\operatorname{dn}hF-\operatorname{dn}(1-h)F}{1-c'}=\frac{CQ-CQ'}{CA-CB}=\cos\chi,\end{aligned}$$


C′
P″
P′
P₂
ψ₁−γ
ψ
B
E
γ
ψ′
L
ψ
ψ₁
P₁
φ′
φ
M
p
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P‴
C
λ
O
F
ψ′
ψ′₁
XII
I
II
III
IIII
V
VI
VII
VIII
IX
X
XI
A

FIG. A.

the geometrical interpretation, since CQ, CQ', $CB = CA\,\mathrm{dn}\,(f,\, 1-f,\, 1)\,F$;

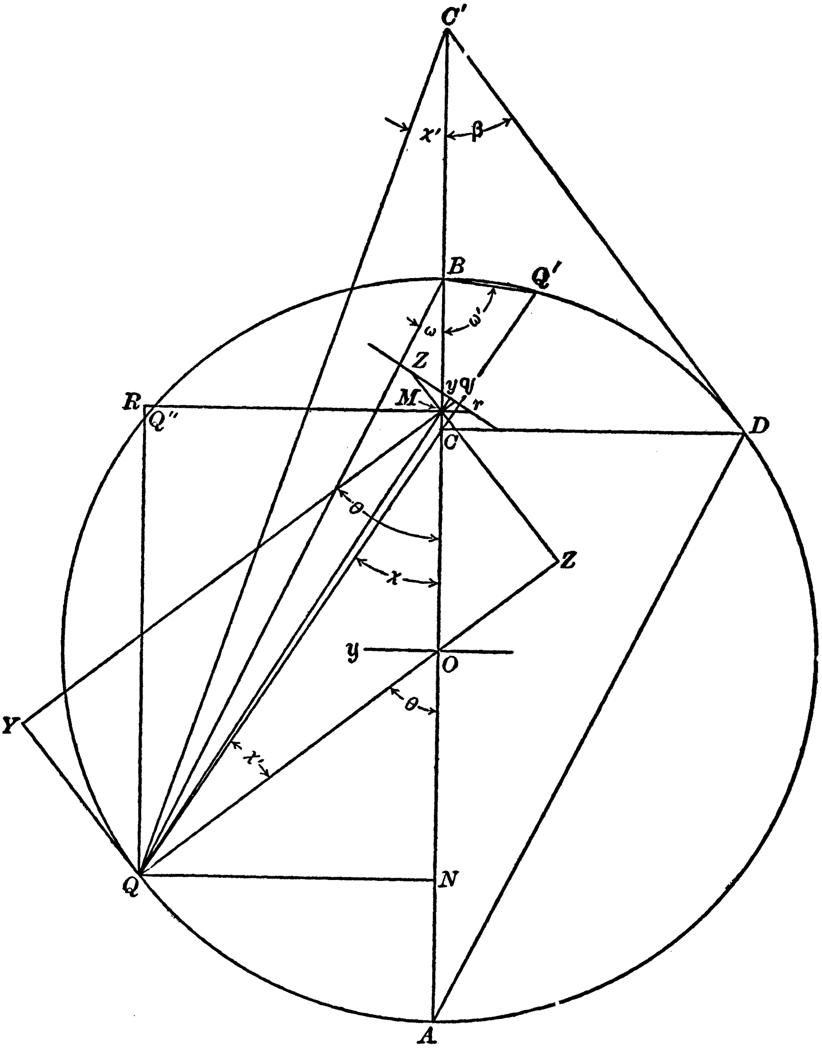

Fig. *B*.

$$\mathrm{dn}\,2hK = \frac{1-(1-c')\,\mathrm{sn}^2 hF}{\mathrm{dn}\,hF} = \frac{\mathrm{dn}^2 hF + c'}{(1+c')\,\mathrm{dn}\,hF} = \frac{QQ'}{AB}. \tag{6}$$

Equation (4) can be written

$$\mathrm{sn}^2\,2hK = \frac{1+c'}{1-c'}\,\frac{1-\mathrm{dn}\,2hF}{1+\mathrm{dn}\,2hF}, \tag{7}$$

and thence

$$\mathrm{dn}\,2hF = \frac{1-k\,\mathrm{sn}^2\,2hK}{1+k\,\mathrm{sn}^2\,2hK}, \tag{8}$$

which is LANDEN's II transformation, in conjunction with

$$(9)\qquad \operatorname{sn} 2hF = \frac{(1+k)\operatorname{sn} 2hK}{1+k\operatorname{sn}^2 2hK},$$

$$(10)\qquad \operatorname{cn} 2hF = \frac{\operatorname{cn} 2hK \operatorname{dn} 2hK}{1+k\operatorname{sn}^2 2hK}.$$

These may also be written, replacing $2h$ by h,

$$(11)\quad \operatorname{sn} hF = \tfrac{1}{2}\sqrt{(1+\operatorname{sn} 2hK\cdot 1+k\operatorname{sn} 2hK)} - \tfrac{1}{2}\sqrt{(1-\operatorname{sn} 2hK\cdot 1-k\operatorname{sn} 2hK)},$$

$$(12)\quad \operatorname{cn} hF = \tfrac{1}{2}\sqrt{(1+\operatorname{sn} 2hK\cdot 1-k\operatorname{sn} 2hK)} + \tfrac{1}{2}\sqrt{(1-\operatorname{sn} 2hK\cdot 1+k\operatorname{sn} 2hK)},$$

$$(13)\quad \begin{aligned} \operatorname{dn} hF &= \frac{\operatorname{dn} 2hK + k\operatorname{cn} 2hK}{1+k} = \frac{1-k}{\operatorname{dn} 2hK - k\operatorname{cn} 2hK} \\ &= \sqrt{\frac{1-k}{1+k}}\sqrt{\frac{\operatorname{dn} 2hK + k\operatorname{cn} 2hK}{\operatorname{dn} 2hK - k\operatorname{cn} 2hK}}. \end{aligned}$$

17. The associated transformation of the complementary period K' or F' can now be written down and interpreted geometrically in fig. A where

$$(1)\quad \psi = \operatorname{am} 2fF',\ \phi = \operatorname{am} fK',\ \phi' = \operatorname{am}(1-f)K',\ 2\phi = \psi + \psi',\ \pi - 2\phi' = \psi - \psi',$$

and as in LANDEN's II transformation in (11) § 16,

$$\sin\phi = \sin\tfrac{1}{2}(\psi+\psi') = \tfrac{1}{2}\sqrt{(1+\sin\psi\cdot 1+\sin\psi')} - \tfrac{1}{2}\sqrt{(1-\sin\psi\ \ 1-\sin\psi')},$$

$$(2)\quad \begin{aligned} \operatorname{sn} fK' &= \tfrac{1}{2}\sqrt{(1+\operatorname{sn} 2fF'\cdot 1+c'\operatorname{sn} 2fF')} - \tfrac{1}{2}\sqrt{(1-\operatorname{sn} 2fF'\cdot 1-c'\operatorname{sn} 2fF')} \\ &= \frac{(1+c')\operatorname{sn} fF'}{1+c'\operatorname{sn}^2 fF'}, \end{aligned}$$

as in (9) § 16 so that $f=0$ leads to (4) § 15;

$$(3)\quad \begin{aligned} \operatorname{cn} fK' &= \tfrac{1}{2}\sqrt{(1+\operatorname{sn} 2fF'\cdot 1-c'\operatorname{sn} 2fF')} + \tfrac{1}{2}\sqrt{(1-\operatorname{sn} 2fF'\ \ 1+c'\operatorname{sn} 2fF')} \\ &= \frac{\operatorname{cn} fF' \operatorname{dn} fF'}{1+c'\operatorname{sn}^2 fF'}, \end{aligned}$$

$$(4)\quad \begin{aligned} \operatorname{dn} fK' &= \frac{1-c'\operatorname{sn}^2 fF'}{1+c'\operatorname{sn}^2 fF'} = \frac{\operatorname{dn} 2fF' + c'\operatorname{cn} 2fF'}{1+c'} \\ &= \frac{1-c'}{\operatorname{dn} 2fF' - c'\operatorname{cn} 2fF'} = \sqrt{\frac{1-c'}{1+c'}}\sqrt{\frac{\operatorname{dn} 2fF' + c'\operatorname{cn} 2fF'}{\operatorname{dn} 2fF' - c'\operatorname{cn} 2fF'}}, \end{aligned}$$

and, conversely,

$$(5)\qquad \operatorname{sn} 2fF' = (1+k)\frac{\operatorname{sn} fK' \operatorname{cn} fK'}{\operatorname{dn} fK'} = \sqrt{\frac{1+k}{1-k}}\sqrt{\frac{1-\operatorname{dn} 2fK'}{1+\operatorname{dn} 2fK'}},$$

derivable from (2, 3, 4), and the geometrical interpretation of

$$(6)\qquad \psi = \tfrac{1}{2}\pi + \phi - \phi', \qquad \sin\psi = \cos(\phi - \phi'), \qquad \cos\psi = \sin(\phi' - \phi);$$

$$(7)\qquad \begin{aligned} \operatorname{cn} 2fF' &= \frac{1 - (1+k)\operatorname{sn}^2 fK'}{\operatorname{dn} fK'} = \frac{\operatorname{dn}^2 fK' - k}{(1-k)\operatorname{dn} fK'} \\ &= \frac{\operatorname{dn} fK' - \operatorname{dn}(1-f)K'}{1-k} = \frac{BP' - BP}{BC' - BC} = \cos\psi, \end{aligned}$$

$$(8)\qquad \begin{aligned} \operatorname{dn} 2fF' &= \frac{1 - (1-k)\operatorname{sn}^2 fK'}{\operatorname{dn} fK'} = \frac{\operatorname{dn}^2 fK' + k}{(1+k)\operatorname{dn} fK'} \\ &= \frac{\operatorname{dn} fK' + \operatorname{dn}(1-f)K'}{1+k} = \frac{BP' + BP}{BC' + BC} = \frac{PP'}{CC'} = \cos\psi', \end{aligned}$$

the geometrical interpretation.

The two circles in figures A and B, linked in perpendicular planes, may be taken to typify the magnetic and electric circuit, associated in a similar manner.

18. In LEGENDRE's notation, the II. E. I.

$$(1)\qquad E(\omega, c) = \int_0^\omega \sqrt{(1 - c^2\sin^2\omega)}\,d\omega = \int \Delta\omega\, d\omega,$$

and the complete II. E. I., or $E(\frac{1}{2}\pi, c)$ or $E(c)$ is denoted by E, and $E(i)$ by E', while as before $E(\frac{1}{2}\pi, k)$ is denoted by H, and $E(k')$ by H'.

In JACOBI's notation, with $\omega = \operatorname{am} hF$, and as employed already in § 3,

$$(2)\qquad \int_0^h \operatorname{dn}^2 hF \cdot F dh = E \operatorname{am} hF = hE + \operatorname{zn} hF.$$

Squaring equation (13) § 16,

$$(3)\qquad \operatorname{dn}^2 hF = \frac{2\operatorname{dn}^2 2hK + 2k\operatorname{cn} 2hK \operatorname{dn} 2hK - k'^2}{(1+k)^2},$$

and then integrating with respect to h, there results

$$(4)\qquad \frac{hE + \operatorname{zn} hF}{F} = \frac{2hH + \operatorname{zn} 2hK + k\operatorname{sn} 2hK - k'^2 hK}{(1+k)^2 K},$$

$$(5)\qquad hE + \operatorname{zn} hF = \frac{2hH - k'^2 hK + \operatorname{zn} 2hK + k\operatorname{sn} 2hK}{1+k},$$

so that, putting $h = 1$,

$$(6)\qquad E = \frac{2H - k'^2 K}{1+k} = \frac{2H}{1+k} - (1-k)K,$$

$$(7)\qquad F-E=2\left(K-\frac{H}{1+k}\right),\qquad E-c'^2F=2\frac{H-(1-k)K}{1+k},$$

and conversely

$$(8)\qquad H=\frac{E+c'F}{1+c'},\qquad HK'=EF'+c'FF',$$

so that, as in Maxwell E. and M. § 701 when corrected,

$$(9)\qquad \frac{2}{c}(F-E)-cF=2\frac{K-H}{\sqrt{k}},$$

$$(10)\qquad 2(F-E)-c^2F=4\frac{K-H}{1+k},$$

$$(11)\qquad (1+c'^2)E-2c'^2F=4\frac{(1+k^2)H-(1-k^2)K}{(1+k)^3}.$$

19. Then

$$(1)\qquad \operatorname{zn} hF=\frac{\operatorname{zn} 2hK+k\operatorname{sn} 2hK}{1+k},$$

which it is convenient to write

$$(2)\qquad F\operatorname{zn} hF=K\operatorname{zn} 2hK+Kk\operatorname{sn} 2hK,$$

and conversely

$$K\operatorname{zn} 2hK=F\operatorname{zn} hF-\tfrac{1}{2}Fc^2\frac{\operatorname{sn} hF\operatorname{cn} hF}{\operatorname{dn} hF}$$

$$(3)\qquad =\tfrac{1}{2}F\operatorname{zn} hF-\tfrac{1}{2}F\operatorname{zn}(1-h)F$$

$$=\tfrac{1}{2}F\operatorname{zn} 2hF-\tfrac{1}{2}Fc^2\frac{\operatorname{sn} 2hF\operatorname{cn} hF}{1+\operatorname{dn} 2hF}.$$

Squaring (4) § 17 and integrating with respect to f, we obtain the remaining relations

$$(4)\qquad H'=\frac{2E'}{1+c'}-(1-c')F',$$

$$(5)\qquad E'=\frac{H'+kK'}{1+k},\qquad E'F=H'K+kKK',$$

$$(6)\qquad \operatorname{zn} fK'=\frac{\operatorname{zn} 2fF'+c'\operatorname{sn} 2fF'}{1+c'},$$

which we write

$$(7)\qquad K\operatorname{zn} fK'=\tfrac{1}{2}F\operatorname{zn} 2fF'+\tfrac{1}{2}Fc'\operatorname{sn} 2fF'$$

$$(8)\qquad K\operatorname{zn}(1-f)K'=-\tfrac{1}{2}F\operatorname{zn} 2fF'+\tfrac{1}{2}Fc'\operatorname{sn} 2fF',$$

and conversely

$$F\operatorname{zn} 2fF'=2K\operatorname{zn} fK'-Kk'^2\frac{\operatorname{sn} fK'\operatorname{cn} fK'}{\operatorname{dn} fK'}$$

$$= K\,\mathrm{zn}\,fK' - K\,\mathrm{zn}\,(1-f)K'$$

$$(9)\qquad = K\,\mathrm{zn}\,2fK' - Kk'^2\frac{\mathrm{sn}\,2fK'\,\mathrm{cn}\,2fK'}{1+\mathrm{dn}\,2fK'}$$

$$= K\,\mathrm{zs}\,2fK' - K\frac{\mathrm{cn}\,2fF'}{\mathrm{sn}\,2fK'},$$

$$(10)\qquad F2\mathrm{n}\,(1-2f)F' = K\,\mathrm{zs}\,(1-2f)K' - Kk\,\mathrm{tn}\,2fK',$$

theorems required in the sequel for the quadric transformation of M, the mutual induction of a helix and a coaxial ring.

20. Integrating (3) § 19 with respect to h,

$$(1)\qquad \log\frac{\Theta 2hK}{\Theta 0K} = 2\log\frac{\Theta hF}{\Theta 0F} + \log\mathrm{dn}\,hF,$$

$$(2)\qquad \frac{\Theta 2hK}{\Theta 0K} = \left(\frac{\Theta hF}{\Theta 0F}\right)^2\mathrm{dn}\,hF = \frac{\Theta hF}{\Theta 0F}\,\frac{\Theta(1-h)F}{\Theta F},$$

the quadric transformation of the theta function.

Similarly by integration of (9), § 19, with respect to f,

$$(3)\qquad \frac{\Theta 2fF'}{\Theta 0F'} = \left(\frac{\Theta fK'}{\Theta 0K'}\right)^2\mathrm{dn}\,fK' = \frac{\Theta fK'}{\Theta 0K'}\,\frac{\Theta(1-f)K'}{\Theta K'}.$$

The same notation is useful for expressing the change in a theta function and eta function from imaginary to real argument and comodulus in the form

$$(4)\qquad \frac{\Theta(2fK'i,\,k)}{\Theta(0,\,k)} = q^{-f^2}\frac{H(1-2f)K'}{HK'},\qquad \frac{H(2fK'i,\,k)}{H(K,\,k)} = iq^{-f^2}\frac{H2fK'}{HK'},$$

where $q = \exp(-\pi K'/K)$. (Jacobi, *Fundamenta nova*, or *Werke*, I, p. 215; Cayley, *Elliptic Functions*, p. 151.)

These transformations may be used for the bisection of the elliptic function;

$$(5)\qquad \left(\frac{\Theta hF}{\Theta 0F}\right)^2 = \frac{\Theta 2hK}{\Theta 0K}\,\frac{\mathrm{dn}\,2hK - k\,\mathrm{cn}\,2hK}{1-k}$$

$$= \frac{1}{1-k}\,\frac{\Theta(1-2h)K}{\Theta 0K} - \frac{k}{1-k}\,\frac{H(1-2h)K}{HK},$$

$$(6)\qquad \left(\frac{\Theta hF}{\Theta F}\right)^2 = \frac{1}{1+k}\,\frac{\Theta(1-2h)K}{\Theta 0K} - \frac{k}{1+k}\,\frac{H(1-2h)K}{HK}.$$

Algebraical form of Landen's transformations.

21. The algebraical equivalent of our first quadric transformation in § 16 is (Tannery and Molk, *Fonctions elliptiques*, formulas XXII, XXIV)

$$(1)\qquad m^2(s-s_1)=\frac{[t-t_1-\surd(t_1-t_2\cdot t_1-t_3)]^2}{4(t-t_1)},$$

$$(2)\qquad m^2(s-s_2)=\frac{[t-t_1+\surd(t_1-t_3\cdot t_2-t_3)]^2}{4(t-t_1},$$

$$(3)\qquad m^2(s-s_3)=\frac{t-t_2\ t-t_3}{4(t-t_1)}=\frac{T}{16(t-t_1)^2},$$

so that the graph of a relation (s, t) is a hyperbola; and

$$(4)\qquad 2m\surd(s_1-s_3)=\surd(t_1-t_3)-\surd(t_1-t_2),$$

$$(5)\qquad 2m\surd(s_2-s_3)=\surd(t_1-t_3)-\surd(t_1-t_2),$$

the equivalent of the modular equation (1) § 15, with k, k' in terms of s_1, s_2, s_3, as in (4) § 2, and c, c' the equivalent for t_1, t_2, t_3. Also

$$(6)\qquad m^2\frac{ds}{dt}=\frac{(t-t_1)^2-(t_1-t_2\cdot t_1-t_3)}{4(t-t_1)^2},$$

$$(7)\qquad m^3\surd S=\frac{(t-t_1)^2-(t_1-t_2\cdot t_1-t_3)}{8(t-t_1)^2}\surd T,$$

where

$$(8)\qquad S=4(s-s_1\cdot s-s_2\cdot s-s_3),\qquad T=4(t-t_1\cdot t-t_2\cdot t-t_3),$$

$$(9)\qquad \frac{ds}{m\surd S}=2\frac{dt}{\surd T}.$$

This algebraical result is obtained by substitution in (4), (5), (6), § 15, of

$$(10)\quad \sin^2\chi=\mathrm{sn}^2\,2hK=\frac{s_1-s_3}{s-s_3},\text{ or }\frac{s-s_3}{s_2-s_3},\ \infty>s>s_1,\text{ or }s_2>s>s_3,$$

$$(11)\quad \sin^2\omega=\ \mathrm{sn}^2hF\ =\frac{t_1-t_3}{t-t_3},\text{ or }\frac{t-t_3}{t_2-t_3},\ \infty>t>t_1,\text{ or }t_2>t>t_3.$$

In the associated complementary transformation of § 18, writing σ and Σ for s and S, τ and T' for t and T, the region $s_1>\sigma>s_2$ is excluded, because $\sigma-s_1$ and $\sigma-s_2$ must have the same sign; but

$$(12)\qquad \sin^2\phi=\mathrm{sn}^2fK'=\frac{s_1-s_3}{s_1-\sigma},\qquad s_3>\sigma>-\infty.$$

$$(13)\qquad \operatorname{sn}^2 fF' = \frac{t_1 - \tau}{t_1 - t_2}, \text{ or } \frac{t_1 - t_3}{t_1 - \tau}, \quad t_1 > \tau > t_2, \text{ or } t_3 > \tau > -\infty,$$

and these substituted in (2), (3), § 17, will satisfy the algebraical relation above.

On applying this algebraical quadric transformation to our III. E. I.,

$$(14)\qquad m^3 \surd(-\Sigma) = \frac{(\tau - t_1)^2 - (t_1 - t_2 \cdot t_1 - t_3)}{8(\tau - t_1)^2} \surd(-T'),$$

$$(15)\qquad m^2(s - \sigma) = \frac{(t - \tau)\left[(t - t_1)(\tau - t_1) - (t_1 - t_2 \cdot t_1 - t_3)\right]}{4(t - t_1)(\tau - t_1)},$$

$$(16)\qquad \frac{2m\surd(-\Sigma)}{s - \sigma} = \frac{\surd(-T')}{t - \tau} - \frac{(t_1 - t_2 \cdot t_1 - t_3)\surd(-T')}{(\tau - t_1)\left[(t - t_1 \cdot \tau - t_1) - (t_1 - t_2 \cdot t_1 - t_3)\right]}$$

$$= \frac{\surd(-T')}{t - \tau} + \frac{\surd(-T')}{u - \tau} + \frac{\surd(-T')}{\tau - t_1},$$

and on putting

$$(17)\qquad t - t_1 = \frac{t_1 - t_2 \cdot t_1 - t_3}{u - t_1}, \qquad \frac{dt}{\surd T} = -\frac{du}{\surd U},$$

$$(18)\qquad (t - t_1 \cdot \tau - t_1) - (t_1 - t_2 \;\; t_1 - t_3) = (t_1 - t_2 \;\; t_1 - t_3)\frac{\tau - u}{u - t_1},$$

so that

$$(19)\qquad \frac{\surd(-\Sigma)}{s - \sigma}\frac{ds}{\surd S} = \frac{\surd(-T')}{t - \tau}\frac{dt}{\surd T} - \frac{\surd(-T')}{u - \tau}\frac{du}{\surd U} - \frac{\surd(-T')}{t_1 - \tau}\frac{dt}{\surd T}.$$

The hyperbolic graph connecting s and t being drawn it shows that as t increases from t_1 to ∞, u diminishes from ∞ to t_1 and s diminishes from ∞ to s_1 and rises again to infinity so that

$$(20)\quad \int_{s_1}^{\infty} \frac{\surd(-\Sigma)}{s - \sigma}\frac{ds}{\surd S} = -\int_{t_1}^{\infty} \frac{\surd(-T')}{t - \tau}\frac{dt}{\surd T} + \sqrt{\left(\frac{\tau - t_1 \cdot \tau - t_3}{t_1 - \tau \cdot t_1 - t_3}\right)} \int \frac{\surd(t_1 - t_3)\,dt}{\surd T},$$

$$(21)\quad 2C(fK') = -2A(fF') + \frac{\operatorname{cn} fF' \operatorname{dn} fF'}{\operatorname{sn} fF'} F = -A(fF') + C(fF').$$

But as t increases from t_3 to t_2, u diminishes from t_2 to t_3, and s increases from s_3 to s_2 and diminishes again to s_3, so that

$$(22)\quad \int_{s_3}^{s_2} \frac{\surd(-\Sigma)}{s - \sigma}\frac{ds}{\surd S} = \int_{t_3}^{t_2} \frac{\surd(-T')}{t - \tau}\frac{dt}{\surd T} + \sqrt{\left(\frac{\tau - t_2 \cdot \tau - t_3}{t_1 - \tau \cdot t_1 - t_3}\right)} \int \frac{\surd(t_1 - t_3)\,dt}{\surd T},$$

$$(23)\quad 2D(fK') = 2B(fF') + \frac{\operatorname{cn} fF' \operatorname{dn} fF'}{\operatorname{sn} fF'} F = B(fF') + D(fF').$$

Making use of the expressions for A, B, C, D in § 5, § 6, equations (21), (23) are equivalent to the single relation

$$(24) \qquad 2K \operatorname{zs} fK' = F \operatorname{zs} fF' + F \operatorname{zn} fF',$$

as before in § 19.

22. For the transformation from t back to s in the III. E. I., we take

$$(1) \qquad \frac{t-t_1}{t_1-\tau} = \frac{\surd(s-s_2)+\surd(s-s_1)}{\surd(s-s_2)-\surd(s-s_1)}\;\frac{\surd(s_1-\sigma)-\surd(s_2-\sigma)}{\surd(s_1-\sigma)+\surd(s_2-\sigma)},$$

$$(2) \qquad \frac{t-\tau}{t_1-\tau} = 2\,\frac{\surd(s-s_2)\surd(s_1-\sigma)-\surd(s-s_1)\surd(s_2-\sigma)}{[\surd(s-s_2)-\surd(s-s_1)]\,[\surd(s_1-\sigma)+\surd(s_2-\sigma)]},$$

$$(3) \qquad \frac{t_1-\tau}{t-\tau} = \frac{\begin{array}{r}[\surd(s_1-\sigma)+\surd(s_2-\sigma)]\,[\surd(s_1-s_2)\surd(s-s_1)] \\ \times\,[s-s_2)\surd(s-\sigma)+\surd(s-s_1)\surd(s_2-\sigma)]\end{array}}{2(s_1-s_2)(s-\sigma)}$$

$$= \tfrac{1}{2} + \frac{\surd(s_1-\sigma\cdot s_2-\sigma)-\surd(s-s_1\cdot s-s_2)}{2(s-\sigma)},$$

$$(4) \qquad \frac{\frac{1}{2}\surd(-T')}{t_1-\tau} = 2\surd(s_3-\sigma),$$

$$(5) \qquad \frac{\frac{1}{2}\surd(-T')}{t-\tau}\,\frac{dt}{\surd T} = \left[\tfrac{1}{2} + \frac{\surd(s_1-\sigma\cdot s_2-\sigma)-\surd(s-s_1\cdot s-s_2)}{2(s-\sigma)}\right]\frac{\surd(s_3-\sigma)\,ds}{\surd S},$$

$$(6) \qquad \int_{t_1,\,t_3}^{\infty,\,t_2} \frac{\frac{1}{2}\surd(-T')}{t-\tau}\,\frac{dt}{\surd T}$$

$$= \sqrt{\frac{s_3-\sigma}{s_1-s_3}}\int_{s_1,\,s_3}^{\infty,\,s_2}\frac{\surd(s_1-s_3)\,ds}{\surd S} + \int\frac{\frac{1}{2}\surd(-\Sigma)}{s-\sigma}\,\frac{ds}{\surd S} - \tfrac{1}{2}\int\frac{\surd(s_3-\sigma)\,ds}{(s-\sigma)\surd(s-s_3)},$$

the last integral being zero as s ranges forward and back again.

According to the region of τ,

$$(7) \qquad t_1 > \tau > t_2, \qquad \text{or} \qquad t_2 > \tau > -\infty,$$

and with

$$s_3 > \sigma > -\infty,$$

the relation (6) is equivalent to

$$(8) \qquad A(fF') = K\frac{\operatorname{cn} fK'}{\operatorname{sn} fK'} - C(fK'),$$

$$(9) \qquad -B(fF') = K\frac{\operatorname{cn} fK'}{\operatorname{sn} fK'} - D(fK'),$$

$$(10) \qquad C(fF') = K\frac{\operatorname{cn} fK'}{\operatorname{sn} fK'} + C(fK'),$$

$$D(fF') = K\frac{\operatorname{cn} fK'}{\operatorname{sn} fK'} + D(fK'), \tag{11}$$

and these reduce to the single relation

$$F \operatorname{zn} fF' = K \operatorname{zs} fK' - K\frac{\operatorname{cn} fK'}{\operatorname{sn} fK'}. \tag{12}$$

23. The algebraical equivalent of the second quadric transformation in (8), (9), (10), § 16, is in a similar way equivalent to a hyperbolic graph between x and s,

$$M^2(x - t_1) = \frac{s - s_1 \cdot s - s_2}{s - s_3} = \frac{\frac{1}{4}S}{(s - s_3)^2}, \tag{1}$$

$$M^2(x - t_2) = \frac{[s - s_3 - \sqrt{(s_1 - s_3 \cdot s_2 - s_3)}]}{s - s_3}, \tag{2}$$

$$M^2(x - t_3) = \frac{[s - s_3 + \sqrt{(s_1 - s_3 \cdot s_2 - s_3)}]^2}{s - s_3}, \tag{3}$$

and as in the modular equation (1) § 15,

$$M\sqrt{(t_1 - t_3)} = \sqrt{(s_1 - s_3)} + \sqrt{(s_2 - s_3)}, \tag{4}$$

$$M\sqrt{(t_1 - t_2)} = \sqrt{(s_1 - s_3)} - \sqrt{(s_2 - s_3)}. \tag{5}$$

Also $Mm = 1$, and

$$M^2\frac{dx}{ds} = \frac{(s - s_3)^2 - (s_1 - s_3 \cdot s_2 - s_3)}{(s - s_3)^2}, \tag{6}$$

$$M^3\sqrt{X} = \frac{(s - s_3)^2 - (s_1 - s_3 \cdot s_2 - s_3)}{(s - s_3)^2}\sqrt{S}, \tag{7}$$

$$\frac{dx}{M\sqrt{X}} = \frac{ds}{\sqrt{S}}, \qquad X = 4(x - t_1 \cdot x - t_2 \cdot x - t_3). \tag{8}$$

The region $t_2 > x > t_3$ is now excluded, because $x - t_2$ and $x - t_3$ must have the same sign; but

$$\sin^2\xi = \operatorname{sn}^2 2hF = \frac{t_1 - t_3}{x - t_3}, \qquad \infty > x > t_1, \tag{9}$$

$$\sin^2\chi = \operatorname{sn}^2 2hK = \frac{s_1 - s_3}{s - s_3} \text{ or } \frac{s - s_3}{s_2 - s_3}, \qquad \infty > s > s_1 \text{ or } s_2 > s > s_3; \tag{10}$$

and these substituted in (8), (9), (10) § 16 will give the algebraical result above.

In the associated complementary transformation with σ, Σ for s, S and x', X' for x, X,

$$(11) \qquad \sin^2\phi = \operatorname{sn}^2 fK' = \frac{s_1 - \sigma}{s_1 - s_2} \text{ or } \frac{s_1 - s_3}{s_1 - \sigma},\ s_1 > \sigma > s_2, \text{ or } s_3 > \sigma > -\infty,$$

$$(12) \qquad \sin^2\psi = \operatorname{sn}^2 2fK' = \frac{t_1 - x'}{t_1 - t_2} \text{ or } \frac{t_1 - t_3}{t_1 - x'},\ t_1 > x' > t_2, \text{ or } t_3 > x' > -\infty,$$

and the algebraical relation is satisfied when these are substituted in (5) § 17.

In the transformation of the III. E. I.,

$$(13) \qquad M^2(x - x') = \frac{(s - \sigma)[(s - s_3 \cdot \sigma - s_3) - (s_1 - s_3 \cdot s_2 - s_3)]}{(s - s_3)(\sigma - s_3)},$$

$$(14) \qquad M^3 \sqrt{(-X')} = \frac{(\sigma - s_3)^2 - (s_1 - s_3 \cdot s_2 - s_3)}{(\sigma - s_3)^2} \sqrt{(-\Sigma)},$$

$$(15) \qquad \begin{aligned} \frac{M\sqrt{(-X')}}{x - x'} &= \frac{(s - s_3)[(\sigma - s_3)^2 - (s_1 - s_3 \cdot s_2 - s_3)]\sqrt{(-\Sigma')}}{(\sigma - s_3)[(s - s_3 \cdot \sigma - s_3) - (s_1 - s_3 \cdot s_2 - s_3)](s - \sigma)} \\ &= \frac{\sqrt{(-\Sigma)}}{s - \sigma} - \frac{\sqrt{(-\Sigma'')}}{s - \sigma''}, \end{aligned}$$

where

$$(16) \qquad \sigma'' - s_3 = \frac{s_1 - s_3\ s_2 - s_3}{\sigma - s_3}, \qquad \sigma'' - s_2 = \frac{s_2 - s_3 \cdot s_1 - \sigma}{\sigma - s_3}, \qquad \sigma'' - s_1 = \frac{s_1 - s_3 \cdot s_2 - \sigma}{\sigma - s_3}.$$

In the arrangement

$$(17) \qquad s_1 > \sigma > s_2, \qquad s_1 > \sigma'' > s_2,$$

$$(18) \qquad \operatorname{sn}^2 f''K' = \frac{s_1 - \sigma''}{s_1 - s_2} = \frac{s_1 - s_3\,\sigma - s_2}{s_1 - s_2\,\sigma - s_3} = \frac{\operatorname{cn}^2 fK'}{\operatorname{dn}^2 fK'},$$

and in the arrangement

$$(19) \qquad s_3 > \sigma > -\infty, \qquad s_3 > \sigma'' > -\infty,$$

$$(20) \qquad \operatorname{sn}^2 f''K' = \frac{s_1 - s_3}{s_1 - \sigma''} = \frac{s_3 - \sigma}{s_2 - \sigma} = \frac{\operatorname{cn}^2 fK'}{\operatorname{dn}^2 fK'},$$

so that

$$(21) \qquad f'' + f = 1.$$

The integral relation

$$(22) \qquad \int \frac{\frac{1}{2}\sqrt{(-X')}}{x - x'} \frac{dx}{\sqrt{X}} = \int \frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s - \sigma} \frac{ds}{\sqrt{S}} - \int \frac{\frac{1}{2}\sqrt{(-\Sigma'')}}{s - \sigma''} \frac{ds}{\sqrt{S}},$$

is now equivalent to

$$(23) \quad A(2fF') = A(fK') - A(1 - f)K', \text{ or } = -B(fK') + B(1 - f)K',$$

implying the relation

$$(24)\qquad F\,\mathrm{zn}\,2fF' = K\,\mathrm{zn}\,fK' - K\,\mathrm{zn}\,(1-f)K',$$

as in (9) § 19; or

$$(25)\quad C(2fF') = C(fK') - C(1-f)K', \text{ or } = D(fK') - D(1-f)K',$$

implying the equivalent to (24), when (5), (7), (8), § 17 are employed:

$$(26)\qquad F\,\mathrm{zs}\,2fF' = K\,\mathrm{zs}\,fK' - K\,\mathrm{zs}\,(1-f)K'.$$

Thus in transformation I, hF and fF' become changed to $2hK$ and fK', and in transformation II, $2hK$ and fK' are changed into $2hF$ and $2fF'$, a successive application being equivalent to a duplication of hF and fF'.

24. To proceed from s to x by means of the II transformation, take

$$(1)\qquad \frac{s-s_3}{\sigma-s_3} = \frac{\surd(x-t_3)+\surd(x-t_2)}{\surd(x-t_3)-\surd(x-t_2)}\cdot\frac{\surd(x'-t_3)-\surd(x'-t_2)}{\surd(x'-t_3)+\surd(x'-t_2)},$$

$$(2)\quad \frac{s-\sigma}{\sigma-s_3} = 2\,\frac{\surd(x-t_2\cdot x'-t_3)-\surd(x-t_3\cdot x'-t_2)}{[\surd(x-t_3)-\surd(x-t_2)]\,[\surd(x'-t_3)+\surd(x'-t_2)]},$$

$$(3)\qquad \frac{\sigma-s_3}{s-\sigma} = \frac{\surd(x'-t_2\cdot x'-t_3)+\surd(x-t_2\cdot x-t_3)}{2(x-x')} - \tfrac{1}{2},$$

$$(4)\qquad \frac{\tfrac{1}{2}\surd(-\Sigma)}{\sigma-s_3} = N\surd(t_1-x'),$$

$$(5)\quad \frac{\surd(-\Sigma)}{s-\sigma}\,\frac{ds}{\surd S} = \left[\frac{\surd(x'-t_2\cdot x'-t_3)+\surd(x-t_2\cdot x-t_3)}{x-x'}-1\right]\frac{\surd(t_1-x')\,dx}{\surd X},$$

$$(6)\quad \int_{s_1,\,s_2}^{\infty,\,s_3}\frac{\surd(-\Sigma)}{s-\sigma}\,\frac{ds}{\surd S} = \int_{t_1}^{\infty}\frac{\tfrac{1}{2}\surd(-X')}{x-x'}\,\frac{dx}{\surd X}$$

$$+\tfrac{1}{2}\int_{t_1}^{\infty}\frac{\surd(t_1-x')\,dx}{(x-x')\surd(x-t_1)} - \sqrt{\frac{t_1-x'}{t_1-t_3}}\int_{t_1}^{\infty}\frac{\surd(t_1-t_3)\,dx}{\surd X},$$

of which the second integral is $\tfrac{1}{2}\pi$, and the third is $Fc'\,\mathrm{sn}\,2fF'$ or $F\,\mathrm{sn}\,2fF'$, according as

$$t_1 > x > t_2 \text{ or } t_3 > x > -\infty;$$

so that in the region $s_1 > \sigma > s_2$, $t_1 > x > t_2$,

$$(7)\qquad 2A(fK') = A(2fF') + \tfrac{1}{2}\pi - Fc'\,\mathrm{sn}\,2fF',$$

$$(8)\qquad 2B(fK') = A(2fF') - \tfrac{1}{2}\pi - Fc'\,\mathrm{sn}\,2fF',$$

equivalent to

$$(9)\qquad 2K\,\mathrm{zn}\,fK' = F\,\mathrm{zn}\,2fF' + Fc'\,\mathrm{sn}\,2fF',$$

as in (7) § 19; and in the region $s_3 > \sigma > -\infty$, $t_3 > x' > -\infty$,

$$(10)\qquad 2C(fK') = C(2fF') - \tfrac{1}{2}\pi - \frac{F}{\operatorname{sn} 2fF'},$$

$$(11)\qquad 2D(fK') = C(2fF') + \tfrac{1}{2}\pi - \frac{F}{\operatorname{sn} 2fF'},$$

equivalent, as in (7) § 19, to

$$(12)\qquad 2K \operatorname{zs} fK' = F \operatorname{zs} 2fF' - F \operatorname{ns} 2fF'.$$

25. Another form of LANDEN's transformation may be added here; put

$$(1)\qquad k = \operatorname{th} \alpha,\ k' = \operatorname{sech} \alpha,\ \text{ then } c' = e^{-2\alpha},$$

$$(2)\qquad c' = \operatorname{th} \delta,\ c = \operatorname{sech} \delta,\ \text{ then } k = e^{-2\delta},$$

and then, as a form of the modular equation of the second order,

$$(3)\qquad (e^{2\alpha} - 1)(e^{2\delta} - 1) = 2, \qquad \operatorname{sh} 2\alpha \operatorname{sh} 2\delta = 1.$$

Now with $\omega = \operatorname{am} hF$, put $\Delta\omega = \operatorname{th} \zeta$,

$$(4)\qquad \begin{aligned} &\sin \omega = \frac{\operatorname{ch} \delta}{\operatorname{ch} \zeta}, \qquad \cos \omega = \frac{\sqrt{(\operatorname{ch}^2 \zeta - \operatorname{ch}^2 \delta)}}{\operatorname{ch} \zeta}, \qquad \operatorname{cn}(1 - h)F = \frac{\operatorname{sh} \delta}{\operatorname{sh} \zeta}, \\ &Fdh = \frac{d\omega}{\Delta\omega} = -\frac{\operatorname{ch} \delta d\zeta}{\sqrt{(\operatorname{ch}^2 \zeta - \operatorname{ch}^2 \delta)}}, \end{aligned}$$

and from (7) § 16,

$$(5)\qquad k \operatorname{sn}^2 hK = e^{-2\zeta}, \qquad \operatorname{sn} hK = e^{-\zeta+\delta}.$$

In the plane xOz of Fig. A, v and w of § 24 are elliptic coördinates, to employ on WEIR's azimuth diagram; while ϕ and k or α are dipolar coördinates, suitable for a stereographic projection or chart, with poles at A and B, in which 2ϕ is the longitude of P, $2\phi'$ of P' both on latitude λ, where

$$(6)\quad \sin \lambda = \operatorname{th} 2\alpha, \qquad \cos \lambda = \operatorname{sech} 2\alpha, \qquad \tan \lambda = \operatorname{sh} 2\alpha, \qquad \tan \tfrac{1}{2}\lambda = \operatorname{th} \alpha = k.$$

We can put, for the conformal representation of the stereographic projection,

$$(7)\qquad z + xi = a \tan(\phi + \alpha i),$$

$$(8)\qquad \frac{x}{a} = \frac{\operatorname{sh} 2\alpha}{\operatorname{ch} 2\alpha + \cos 2\phi}, \qquad \frac{z}{a} = \frac{\sin 2\phi}{\operatorname{ch} 2\alpha + \cos 2\phi},$$

$$(9)\qquad \frac{PA, PB}{a} = \frac{r_1, r_2}{a} = \frac{e^{\pm\alpha}}{\sqrt{[\tfrac{1}{2}(\operatorname{ch} 2\alpha + \cos 2\phi)]}},$$

and if OPP_2 is a straight line

$$(10)\qquad \phi + \phi_2 = \tfrac{1}{2}\pi.$$

But if APP'' is a straight line

$$(11)\qquad \psi + \psi'' = \pi,$$

and ψ, ψ' are biangular coördinates of P, so that $\psi'' = \operatorname{am} 2(1-f)F'$, with the value of ψ, ϕ, ϕ' in (1) § 17; also

$$(12)\qquad BP = \frac{a(1-k)}{\operatorname{dn} fK'}, \qquad AP = \frac{a(1+k)}{\operatorname{dn} fK'},$$

$$(13)\qquad BP' = BP'' = \frac{a(1-k)}{\operatorname{dn}(1-f)K'}, \qquad AP' = AP'' = \frac{a(1+k)}{\operatorname{dn}(1-f)K'},$$

$$(14)\qquad OC\cdot OC' = a^2 = OB^2, \qquad PB\cdot BP' = OB^2\frac{(1-k)^2}{k},$$

$$(15)\qquad OC' = \frac{a}{k}, \qquad OC = ak, \qquad C'B = a\left(\frac{1}{k}-1\right), \qquad BC = a(1-k).$$

Coefficient of mutual induction of two coaxial circles.

26. Apply the preceding method to Maxwell's § 701 (E. and M.). The mutual induction of a short element of wire at P perpendicular to the plane APB of fig. A and of a circular wire on a diameter AB in the perpendicular plane of fig. B, is, for the arc AQ where $AOQ = \theta$,

$$(1)\qquad \int \frac{\cos\epsilon}{PQ} ds = \int_0^\theta \frac{a\cos(\pi-\theta)\,d\theta}{\sqrt{(a^2+2aA\cos\theta+A^2+b^2)}},$$

an integral composed of incomplete I and II. E. I.'s: and this must be integrated round the circle AQB to obtain the vector potential G at P perpendicular to the plane APB; and then multiplied by $2\pi A$ to obtain M for the two circles in parallel planes.

According to our method of leaving the algebraical expression intact, we first substitute

$$(2)\qquad PQ^2 = a^2 + 2aA\cos\theta + A^2 + b^2 = m^2(t_1 - t),$$

and supposing $t = t_3$ and t_2 at A and B, where $\theta = 0$ and π,

$$(3)\qquad PA^2 = r_1^2 = (a+A)^2 + b^2 = m^2(t_1-t_3), \qquad PB^2 = r_2^2 = (a-A)^2 + b^2 = m^2(t_1-t_2),$$

$$(4)\qquad \begin{aligned} PQ^2 - r_2^2 &= 2aA(1+\cos\theta) = m^2(t_2-t), \\ r_1^2 - PQ^2 &= 2aA(1-\cos\theta) = m^2(t-t_3), \end{aligned}$$

so that

$$(5)\qquad 4aA\cos\theta = m^2(t_2+t_3-2t), \qquad r_1^2 - r_2^2 = 4aA = m^2(t_2-t_3),$$

$$(6)\qquad \frac{r_2}{r_1} = \sqrt{\frac{t_1-t_2}{t_1-t_3}} = c', \qquad \frac{2\sqrt{(aA)}}{r_1} = \sqrt{\frac{t_2-t_3}{t_1-t_3}} = c;$$

(7) $\theta = 2\omega = 2\tan^{-1}\sqrt{\dfrac{t-t_3}{t_2-t}},\quad d\theta = \dfrac{dt}{\sqrt{(t_2-t\cdot t-t_3)}},\quad \dfrac{d\theta}{PQ} = \dfrac{2}{r_1}\dfrac{\sqrt{(t_1-t_3)}dt}{\sqrt{T}},$

(8)
$$M = \int_0^{2\pi}\frac{2\pi aA\cos(\pi-\theta)d\theta}{PQ} = 2\pi r_1\int_{t_3}^{t_2}\frac{2t-t_2-t_3}{\sqrt{(t_1-t_3)}}\frac{dt}{\sqrt{T}}$$
$$= 4\pi r_1\int\frac{t-t_3}{\sqrt{(t_1-t_3)}}\frac{dt}{\sqrt{T}} - 2\pi r_1\frac{t_2-t_3}{t_1-t_3}\int\frac{\sqrt{(t_1-t_3)}dt}{\sqrt{T}}.$$

and then by (10) § 3 and (1), (4) § 2,

(9) $$M = 2\pi r_1[2(F-E)-c^2F],$$

as in MAXWELL, with the original sign changed; and by the quadric transformation employing (10) § 18

(10) $$M = 8\pi r_1\frac{K-H}{1+k} = 4\pi(r_1+r_2)(K-H) = 8\pi\sqrt{(aA)}\frac{K-H}{\sqrt{k}},$$

doubling MAXWELL's original result, and to be divided by 10^9 to bring it to henries.

MAXWELL's modular angle γ is now seen to be AEB in fig. A where AE is the tangent from A to the circle on the diameter CC', where PC, PC' are the bisectors of the angle APB; and then BE is at right angles to AB, so that A, B being limiting points of this circle

(11) $$\cos\gamma = \frac{EB}{EA} = \frac{PB}{PA} = \frac{r_2}{r_1} = c'.$$

Putting $k = \sin\beta$, so that β is the modular angle of the period K and K',

(12) $$\sin\beta = \frac{r_1-r_2}{r_1+r_2} = \frac{AC-CB}{AC+CB} = \frac{OC}{OA} = \frac{AC}{AC'},$$

so that β is the angle $OC'D$ in fig. B where CD is perpendicular to AB, and $C'D$ the tangent at D.

27. A geometrical interpretation can be given in LANDEN's manner of these transformations in the plane AQB of the circular disc or wire in fig. B, as well as in the perpendicular plane APB of fig. A, in which PC, PC', the bisectors of the angle APB, are axes of the elliptic cone whose vertex is P and circular base AQB.

The point F in fig. A where PC meets the axis of the circle AQB is the center of a sphere of which C and P are limiting points, so that PQ/QC is a constant ratio.

Now, ω denoting the angle ABQ in fig. B,

(1) $$PQ^2 = r_1^2\cos^2\omega + r_2^2\sin^2\omega,$$

and at D, where PDC' is the tangent plane of the cone,

$$\begin{aligned}(2)\qquad &\cos 2\omega = -\frac{OC}{OB} = -\frac{r_1 - r_2}{r_1 + r_2},\\ &\cos^2\omega = \frac{r_2}{r_1 + r_2}, \qquad \sin^2\omega = \frac{r_1}{r_1 + r_2},\end{aligned}$$

$$(3)\quad ABD = \operatorname{am}\tfrac{1}{2}F, \qquad PD^2 = r_1 r_2, \qquad CD^2 = AC\cdot CB = \frac{4r_1 r_2 a^2}{(r_1 + r_2)^2},$$

so that, round the circle AQB,

$$(4)\qquad \frac{PQ}{QC} = \frac{PD}{DC} = \frac{r_1 + r_2}{2a},$$

and so also

$$(5)\qquad \frac{PQ}{QC'} = \frac{r_1 - r_2}{2a}, \qquad \frac{QC}{QC'} = \frac{r_1 - r_2}{r_1 + r_2} = k.$$

On fig. B the angles QCA, $QC'A$ or CQO are denoted by χ, χ', and QBA, $Q'BA$ by ω, ω'; and we put, as in § 16,

$$(6)\qquad \chi = \operatorname{am} 2hK, \qquad \omega = \operatorname{am} hF, \qquad \omega' = \operatorname{am}(1 - h)F,$$

$$(7)\qquad PQ = r_1 \operatorname{dn} hF, \qquad CQ = \frac{2ar_1}{r_1 + r_2}\operatorname{dn} hF = CA \operatorname{dn} hF,$$

and then, since $OQC = \chi'$, $OQC' = \chi$, $\sin\chi' = k\sin\chi$, $\cos\chi' = \operatorname{dn} 2hK$,

$$(8)\qquad CQ,\ CQ' = a\cos\chi' \pm ak\cos\chi = a(\operatorname{dn} 2hK \pm k \operatorname{cn} 2hK),$$

$$(9)\qquad PQ,\ PQ' = \tfrac{1}{2}(r + r_2)(\cos\chi' \pm k\cos\chi), \qquad PQ\cdot PQ' = r_1 r_2,$$

$$(10)\qquad QQ' = 2a\cos\chi' = 2a\operatorname{dn} 2hK,$$

$$(11)\qquad \sin\chi = \frac{a\sin\theta}{CQ} = (r_1 + r_2)\frac{\sin\omega\cos\omega}{PQ},$$

$$(12)\qquad (r_1 + r_2)^2 \operatorname{cec}^2\chi = r_1^2 \operatorname{cec}^2\omega + r_2^2 \sec^2\omega,$$

$$(13)\qquad (r_1 + r_2)\frac{\cos\chi}{\sin\chi} = r_1\cot\omega - r_2\tan\omega,$$

$$(14)\qquad (r_1 + r_2)\frac{\cos\chi'}{\sin\chi} = r_1\cot\omega + r_2\tan\omega,$$

$$(15)\qquad \sin^2\omega = \tfrac{1}{2}(1 - \cos\theta) = \tfrac{1}{2}(1 + \sin\chi\sin\chi' - \cos\chi\cos\chi'),$$

$$(16)\qquad \sin\omega = \tfrac{1}{2}\sqrt{(1 + \sin\chi\cdot 1 + \sin\chi')} - \tfrac{1}{2}\sqrt{(1 - \sin\chi\cdot 1 - \sin\chi')},$$

etc., the geometrical equivalent of the LANDEN transformation in § 16.

28. In the transformation of M to MAXWELL's second form in § 702, we notice that, from elementary geometrical considerations,

$$(1)\quad \frac{a d\theta}{CQ} = \frac{d\chi}{\cos\chi'}, \quad \frac{a d\theta}{PQ} = \frac{2a}{r_1 + r_2}\frac{d\chi}{\cos\chi'}, \quad \sin\chi' = k\sin\chi, \quad \cos\chi' = \Delta\chi,$$

$$M = \int_0^{2\pi} \frac{-2\pi a A\cos\theta d\theta}{PQ} = \frac{-8\pi a A}{r_1 + r_2}\int_0^{\pi} \cos(\chi + \chi')\frac{d\chi}{\cos\chi'}$$

$$(2)\quad = \frac{8\pi a A}{r_1 + r_2}\int\left(\frac{k\sin^2\chi}{\cos\chi'} - \cos\chi\right)d\chi$$

$$= \frac{8\pi a A}{r_1 - r_2}\int_0^{\pi}\left(\frac{1 - \Delta^2\chi}{\Delta\chi} - k\cos\chi\right)d\chi = 4\pi(r_1 + r_2)(K - H).$$

Hence another method in place of MAXWELL's § 702 for drawing the lines of magnetic force $M =$ constant, employing WEIR's Azimuth Diagram covered by confocal conics for which $r_1 \pm r_2$ is constant.

Denoting WEIR's hour angle by α and latitude angle by λ, then

$$(3)\quad \sin\alpha = \frac{r_1 - r_2}{2a}, \quad \cos\lambda = \frac{2a}{r_1 + r_2}, \quad k = \sin\alpha\cos\lambda, \quad \frac{M}{8\pi a} = \frac{K - H}{\cos\lambda}.$$

Supposing the curved line CPC' in fig. A to represent for a moment a line of magnetic force of constant M, starting from C where $\lambda = 0$, and orthogonal to the lines of constant Ω, such as those shown radiating from A,

$$(4)\quad k = \sin\alpha, \quad \frac{M}{8\pi a} = K - H.$$

At C' where $\alpha = 90°$,

$$(5)\quad k = \cos\lambda, \quad \frac{M}{8\pi a} = \frac{K - H}{k}.$$

To find an intermediate point P on CPC' assume an arbitrary value of k and $K - H$, less than the value at C, and calculate λ and α from the relation

$$(6)\quad \cos\lambda = (K - H) \div \frac{M}{8\pi a},$$

$$(7)\quad \sin\alpha = \frac{k}{\cos\lambda} = \frac{M}{8\pi a} \div \frac{K - H}{k}.$$

We thus require a table of $K - H$ and $(K - H)/k$, say for every five degrees of the modular angle $\theta = \sin^{-1} k$; the work of redrawing MAXWELL's figure XVIII of the lines of magnetic force for a circular current is being carried out on this method.

The meridians and parallels of latitude of a stereographic projection on fig. A, with poles at A and B, will map out the electromagnetic field of a straight current through A and the return through B perpendicular to the plane, by means of the longitude for magnetic potential, and then $M = 2\alpha = \log(r_1/r_2)$.

29. Putting $\cos\theta = c$, the algebraical form of the quadric substitution is

$$(1) \qquad y \text{ or } y - y_3 = m^2(s - s_3) = \frac{a^2 \sin^2\theta}{PQ^2} = \sin^2 QPN,$$

of which the graph is a hyperbola in the coördinates (c, y);

$$(2) \qquad y_1 - y = m^2(s_1 - s) = \left[\frac{a(c - c_1)}{PQ}\right]^2,$$

$$(3) \qquad y_2 - y = m^2(s_2 - s) = \left[\frac{a(c - c_2)}{PQ}\right]^2,$$

because

$$(4) \qquad a^2c^2 + 2aAyc + (a^2 + A^2 + b^2)y - a^2 = 0,$$

$$(5) \qquad (ac + Ay)^2 = A^2(y_1 - y)(y_2 - y),$$

$$(6) \quad y_1 = \left(\frac{r_1 + r_2}{2A}\right)^2 = \left(\frac{2a}{r_1 - r_2}\right)^2, \qquad c_1 = -\frac{Ay_1}{a} = -\frac{r_1 + r_2}{r_1 - r_2} = -\sqrt{\frac{y_1}{y_2}},$$

$$(7) \quad y_2 = \left(\frac{r_1 - r_2}{2A}\right)^2 = \left(\frac{2a}{r_1 + r_2}\right)^2, \qquad c_2 = -\frac{Ay_2}{a} = -\frac{r_1 - r_2}{r_1 + r_2} = -\sqrt{\frac{y_2}{y_1}},$$

$$(8) \qquad y_1y_2 = \left(\frac{a}{A}\right)^2, \qquad c_1c_2 = 1,$$

$$(9) \qquad c = \frac{-y + \sqrt{(y_1 - y \cdot y_2 - y)}}{\sqrt{(y_1y_2)}},$$

$$c - c_1 = \frac{\sqrt{(y_1 - y)}[\sqrt{(y_1 - y)} + \sqrt{(y_2 - y)}]}{\sqrt{(y_1y_2)}},$$

$$c - c_2 = \frac{\sqrt{(y_2 - y)}[\sqrt{(y_1 - y)} + \sqrt{(y_2 - y)}]}{\sqrt{(y_1y_2)}},$$

$$\frac{PQ}{a} = \frac{\sqrt{(y_1 - y)} + \sqrt{(y_2 - y)}}{\sqrt{(y_1y_2)}}, \qquad \frac{CQ}{a} = \frac{\sqrt{(y_1 - y)} + \sqrt{(y_2 - y)}}{\sqrt{y_1}},$$

$$\frac{PQ}{A} = \sqrt{(y_1 - y)} + \sqrt{(y_2 - y)},$$

$$\sin\theta = \frac{\sqrt{y}\,[\sqrt{(y_1 - y)} - \sqrt{(y_2 - y)}]}{\sqrt{(y_1y_2)}},$$

$$\tan\tfrac{1}{2}\theta = \frac{(\sqrt{y_1} + \sqrt{y_2})\sqrt{y}}{\sqrt{y_2}\sqrt{(y_1 - y)} + \sqrt{y_1}\sqrt{(y_2 - y)}}.$$

Taking logarithmic differentials of $\tan \frac{1}{2}\theta$ and $\sin \theta$,

$$(10) \quad \frac{d\theta}{\sin\theta} \quad \frac{\sqrt{(y_1 y_2)}\,dy}{2y\sqrt{(y_1 - y \cdot y_2 - y)}}, \qquad \cos\theta \frac{d\theta}{\sin\theta} = \frac{dy}{2y} + \frac{dy}{2\sqrt{(y_1 - y \cdot y_2 - y)}},$$

and multiplying by $\sqrt{y} = a \sin\theta / PQ$,

$$(11) \qquad \frac{a\,d\theta}{PQ} = \frac{\sqrt{(y_1 y_2)}\,dy}{\sqrt{Y}}, \qquad \frac{a\cos\theta\,d\theta}{PQ} = \frac{dy}{2\sqrt{y}} + \frac{y\,dy}{\sqrt{Y}},$$

$$(12) \qquad Y = 4y \cdot y_1 - y \cdot y_2 - y;$$

and thence the integrals in the algebraical form for determining MAXWELL's second expression for M in terms of K and H, noticing that y increases from zero to y_2 and back again to zero, as θ increases from 0 to π.

With the transformation of (4) § 25

$$\frac{M}{4\pi\sqrt{(aA)}} = \int^{\infty} \frac{2\,\mathrm{ch}^2\,\delta - \mathrm{ch}^2\,\zeta}{\mathrm{ch}^2\,\zeta} \frac{d\zeta}{\sqrt{(\mathrm{ch}^2\,\zeta - \mathrm{ch}^2\,\delta)}}$$

$$(13) \qquad = \sqrt{2} \int \frac{2\,\mathrm{ch}\,2\delta - \mathrm{ch}\,2\zeta + 1}{\mathrm{ch}\,2\zeta + 1} \frac{d\zeta}{\sqrt{(\mathrm{ch}\,2\zeta - \mathrm{ch}\,2\delta)}}$$

$$= \sqrt{2} \cdot \int^{\infty} \left[\frac{e^{-2\zeta}}{\sqrt{(\mathrm{ch}\,2\zeta - \mathrm{ch}\,2\delta)}} + \tfrac{1}{2} \frac{d}{d\zeta} \frac{\sqrt{(\mathrm{ch}\,2\zeta - \mathrm{ch}\,2\delta)}}{e^{2\zeta} + 1} \right] d\zeta,$$

of which the second term vanishes.

Components of electromagnetic force of a circular current.

30. We can now express the components of M, and also of the magnetic potential Ω, in terms of E and F, or H and K, so as to be able to select the simplest form. Thus

$$\frac{dM}{db} = \int_0^{2\pi} \frac{2\pi a A b \cos\theta\,d\theta}{PQ^3} = -\frac{2\pi b}{m} \int_{t_3}^{t_2} \frac{2t - t_2 - t_3}{t_1 - t} \frac{dt}{\sqrt{T}}$$

$$(1) \qquad = -\frac{2\pi b}{r_1} \int \left(\frac{t_1 - t_3 + t_1 - t_2}{t_1 - t} - 2 \right) \frac{\sqrt{(t_1 - t_3)}\,dt}{\sqrt{T}}$$

$$= -\frac{2\pi b}{r_1} \left(\frac{1 + c'^2}{c'^2} E - 2F \right) = -\frac{2\pi b r_1}{r_2^2} \left[(1 + c'^2) E - 2c'^2 F \right],$$

by a theorem analogous to those in § 3,

$$(2) \qquad \int_{t_3}^{t} \frac{t_1 - t_2}{t_1 - t} \frac{\sqrt{(t_1 - t_3)}\,dt}{\sqrt{T}} = \int_0^h \mathrm{dn}^2 (1 - h) F \cdot F\,dh = hE - \mathrm{zn}\,(1 - h) F.$$

Or, with

$$\cos\theta = \cos(\chi + \chi') = \frac{(\cos\chi' + k\cos\chi)(\cos\chi - k\cos\chi')}{k'^2}, \tag{3}$$

$$PQ = \tfrac{1}{2}(r_1 + r_2)(\cos\chi' + k\cos\chi), \qquad d\theta = (\cos\chi' + k\cos\chi)\frac{d\chi}{\cos\chi'},$$

$$\begin{aligned}\frac{dM}{db} &= \frac{32\pi aAb}{(r_1+r_2)^3 k'^2}\int_0^\pi \frac{\cos\chi - k\cos\chi'}{\cos\chi' + k\cos\chi}\,\frac{d\chi}{\cos\chi'}\\ &= \frac{32\pi aAb}{(r_1+r_2)^3 k'^4}\int [(1+k^2)\cos\chi\cos\chi' - k(\cos^2\chi + \cos^2\chi')]\,\frac{d\chi}{\cos\chi'}\\ &= -\frac{32\pi aAb}{(r_1+r_2)^3 kk'^4}\int [(1+k^2)\cos^2\chi' - (1-k^2)]\,\frac{d\chi}{\cos\chi'}\\ &= -\frac{16\pi b}{r_1+r_2}\,\frac{(1+k^2)H - (1-k^2)K}{k'^4}.\end{aligned} \tag{4}$$

Next

$$\begin{aligned}A\frac{dM}{dA} &= M + \int_0^{2\pi}\frac{2\pi aA\cos\theta(A^2 + aA\cos\theta)\,d\theta}{PQ^3}\\ &= M + \frac{A^2}{b}\frac{dM}{db} + \int\frac{2\pi(aA\cos\theta)^2 d\theta}{PQ^3},\end{aligned} \tag{5}$$

and similarly

$$a\frac{dM}{da} = M + \frac{a^2}{b}\frac{dM}{db} + \int\frac{2\pi(aA\cos\theta)^2 d\theta}{PQ^3}, \tag{6}$$

of which the last integral C is given by

$$\begin{aligned}C &= \tfrac{1}{2}\pi m\int_{t_2}^{t_2}\frac{(t_2+t_3-2t)^2}{t_1-t}\,\frac{dt}{\sqrt{T}}\\ &= \tfrac{1}{2}\pi r_1\left[\frac{(2t_1-t_2-t_3)^2}{t_1-t_2\;\,t_1-t_3}\int\frac{t_1-t_2}{t_1-t}\,\frac{\sqrt{(t_1-t_3)}\,dt}{\sqrt{T}}\right.\\ &\qquad\left. - 4\,\frac{2t_1-t_2-t_3}{t_1-t_3}\int\frac{\sqrt{(t_1-t_3)}\,dt}{\sqrt{T}} + 4\int\frac{t_1-t}{\sqrt{(t_1-t_3)}}\,\frac{dt}{\sqrt{T}}\right]\\ &= \tfrac{1}{2}\pi r_1\left[\frac{(1+c'^2)^2}{c'^2}E - 4(1+c'^2)F + 4E\right] = \tfrac{1}{2}\pi r_1\frac{c^4}{c'^2}E - M,\end{aligned} \tag{7}$$

by preceding theorems on the I. and II. E. I., so that

$$A\frac{dM}{dA} = \tfrac{1}{2}\pi r_1\frac{c^4}{c'^2}E + \frac{A^2}{b}\frac{dM}{db}, \tag{8}$$

$$(9)\qquad a\frac{dM}{da}=\tfrac{1}{2}\pi r_1\frac{c^4}{c'^2}E+\frac{a^2}{b}\frac{dM}{db},$$

and the relation

$$(10)\qquad a\frac{dM}{da}+A\frac{dM}{dA}+b\frac{dM}{db}=M$$

is satisfied, required in consequence of M being a homogeneous function of a, A, b, of the first degree. Otherwise

$$C=\frac{32\pi a^2A^2}{(r_1+r_2)^3}\int_0^\pi\frac{(\cos\chi-k\cos\chi')^2}{k'^4}\frac{d\chi}{\cos\chi'}$$

$$(11)\qquad \begin{aligned}&=\frac{32\pi a^2A^2}{(r_1+r_2)^3}\int\frac{(1+k^4)\cos^2\chi'-(1-k^2)-2k^3\cos\chi\cos\chi'}{k^2k'^4}\frac{d\chi}{\cos\chi'}\\&=\frac{64\pi a^2A^2}{(r_1+r_2)^3}\frac{(1+k^4)H-(1-k^2)K}{k^2k'^4}\\&=4\pi(r_1+r_2)\frac{(1+k^4)H-(1-k^2)K}{k'^4},\end{aligned}$$

$$(12)\qquad M+C=4\pi(r_1+r_2)\frac{k^2}{k'^4}(2H-k'^2K).$$

31. If Ω denotes the magnetic potential or conical angle at P of the circular ring AQB, the magnetic components are (E. and M., § 703),

$$(1)\qquad \frac{d\Omega}{dA}=\frac{1}{2\pi A}\frac{dM}{db}=-\frac{4ab}{r_1^3}\frac{(1+c'^2)E-2c'^2F}{c^2c'^2}$$

or

$$-\frac{32ab}{(r_1+r_2)^3}\frac{(1+k^2)H-(1-k^2)K}{kk'^4},$$

$$(2)\qquad \frac{d\Omega}{db}=-\frac{1}{2\pi A}\frac{dM}{dA}=-\frac{M+C}{2\pi A^2}-\frac{1}{2\pi b}\frac{dM}{db}=-\frac{4a^2}{r_1^3}\frac{E}{c'^2}+\frac{1}{r_1}\frac{(1+c'^2)E-2c'^2F}{c'^2}$$

or

$$-\frac{32a^2}{(r_1+r_2)^3}\frac{2H-k'^2K}{k'^4}+\frac{3}{r_1+r_2}\frac{(1+k^2)H-(1-k^2)K}{k'^4}.$$

Also, from the homogeneity of Ω, of zero dimension,

$$(3)\qquad a\frac{d\Omega}{da}=-A\frac{d\Omega}{dA}-b\frac{d\Omega}{db}=\frac{b}{2\pi A^2}(M+C)=\frac{4a^2b}{r_1^3}\frac{E}{c'^2}$$

or

$$\frac{32a^2b}{(r_1+r_2)^3}\frac{2H-k'^2K}{k'^4}.$$

In the neighborhood of the axis, where A, $r_1 - r_2$, c, and k are small,

$$(4)\qquad F = \tfrac{1}{2}\pi\left(1 + \frac{c^2}{4} + \frac{9c^4}{64}\cdots\right), \qquad E = \tfrac{1}{2}\pi\left(1 - \frac{c^2}{4} - \frac{3c^4}{64}\cdots\right),$$

$$(5)\qquad F - E = \tfrac{1}{2}\pi(\tfrac{1}{2}c^2 + \tfrac{3}{16}c^4\cdots), \qquad E - c'^2F = \tfrac{1}{2}\pi(\tfrac{1}{2}c^2 + \tfrac{13}{16}c^4\cdots),$$

$$(6)\qquad (1+c'^2)E - 2c'^2F = \frac{3\pi c^4}{16},$$

$$(7)\qquad \frac{2H - k'^2K}{k'^4} = \tfrac{1}{2}\pi\left(1 + \frac{9k^2}{4}\cdots\right),$$

$$(8)\qquad \frac{dM}{db} = -\frac{6\pi^2a^2A^2b}{r_1^5}, \quad \text{or} \quad -\frac{192\pi^2a^2A^2b}{(r_1+r_2)^5},$$

$$(9)\qquad M + C = \tfrac{1}{2}\pi r_1 \frac{16a^2A^2}{r_1^4}\tfrac{1}{2}\pi\left(1 + \frac{3c^2}{4}\cdots\right) = \frac{4\pi^2a^2A^2}{r_1^3}\left(1 + \frac{3aA}{r_1^2}\right)$$

or

$$= 4\pi(r_1 + r_2)\frac{16a^2A^2}{(r_1+r_2)^4}\tfrac{1}{2}\pi(1 + \tfrac{9}{4}k^2) = \frac{32\pi^2a^2A^2}{(r_1+r_2)^3}\left[1 + \frac{36a^2A^2}{(r_1+r_2)^4}\right],$$

$$(10)\qquad \frac{d\Omega}{dA} = \frac{3\pi a^2Ab}{r_1^5}, \quad \text{or} \quad \frac{96\pi a^2Ab}{(r_1+r_2)^5},$$

$$(11)\qquad \frac{d\Omega}{db} = \frac{2\pi a^2}{r_1^3} + \frac{6\pi a^3A}{r_1^5}, \quad \text{or} \quad \frac{16\pi a^2}{(r_1+r_2)^3}\left[1 + \frac{36a^2A^2}{(r_1+r_2)^4}\right] - \frac{96\pi a^2A^2}{(r_1+r_2)^5}$$

On the axis itself, $A = 0$,

$$(12)\qquad \frac{d\Omega}{dA} = 0, \qquad \frac{d\Omega}{db} = \frac{2\pi a^2}{r_1^3}.$$

A simpler resolution of the magnetic force is into components G_1 and G_2, perpendicular to PA and PB, and we find (A. Russell, Philosophical Magazine, April, 1907),

$$(13)\qquad G_1 = \frac{F-E}{A} = \frac{2}{A}\left(K - \frac{H}{1+k}\right), \qquad G_2 = \frac{E - c'^2F}{c'A} = \frac{2}{A}\left(\frac{H}{1-k} - K\right),$$

and on the axis $A = 0$, $G_1 = G_2 = \pi a/r_1^2$, but G_1 and G_2 are infinite along the wire.

A similar resolution of the gravitational force of the potential $\int a d\theta / PQ$ of the circular ring AB will show that it can be resolved into the components $2G_1$ and $2G_2$, but now in the direction of PA and PB.

Galvanometer constant of a circular coil.

32. To calculate the galvanometer constant G at a point on the axis of a coil of wire of n turns in the form of a ring of given cross section α, we have to evaluate the integral

$$(1) \qquad G=\frac{n}{\alpha}\int\int\frac{2\pi x^2\,dxdy}{(x^2+y^2)^{\frac{3}{2}}}=\frac{2\pi n}{\alpha}\int\frac{ydx}{(x^2+y)^{\frac{1}{2}}},$$

taken over the area or round the perimeter.

For a circular section AQB, with the axis of the ring through C' in fig. B,

$$(2) \qquad G=\frac{2\pi n}{\pi a^2}\int_0^{2\pi}\frac{a^2\sin^2\theta d\theta}{C'Q}=\int_0^{2\pi}\frac{2n\sin^2\theta d\theta}{C'Q},$$

and putting as before in § 26, with $OC'=a/k$,

$$(3) \quad C'Q^2=m^2(t_1-t),\qquad C'Q^2-C'B^2=\frac{2a^2}{k}(1+\cos\theta)=m^2(t_2-t),$$

$$C'A^2-C'Q^2=\frac{2a^2}{k}(1-\cos\theta)=m^2(t-t_3),$$

we have

$$(4) \quad m^2(t_1-t_3)=C'A^2=a^2\left(\frac{1}{k}+1\right)^2,\qquad m^2(t_1-t_2)=C'B^2=a^2\left(\frac{1}{k}-1\right),$$

$$(5) \qquad c'=\sqrt{\frac{t_1-t_2}{t_1-t_3}}=\frac{1-k}{1+k},$$

$$G=\frac{2nk^2m^3}{a^4}\int_{t_3}^{t_2}(t_2-t\cdot t-t_3)\frac{dt}{\sqrt{T}}$$

$$(6) \qquad \begin{aligned} &=\frac{2nk^2m^3}{3a^4}\int\left[-2\sqrt{T}\frac{d}{dt}\sqrt{T}+(t_1-t_3+t_1-t_2)(t_1-t)-2(t_1-t_2\cdot t_1-t_3)\right]\frac{dt}{\sqrt{T}}\\ &=\frac{2nk^2m^3}{3a^4}\left[(t_1-t_3+t_1-t_2)\sqrt{(t_1-t_3)}E-2(t_1-t_2)\sqrt{(t_1-t_3)}F\right]\\ &=\frac{32n}{3AC'}\,\frac{(1+c'^2)E-2c'^2F}{c^4}.\end{aligned}$$

The reduction can also be carried out with the functions K and H, by means of the angles χ and χ', and

$$(7) \qquad C'Q = \frac{a}{k}\cos\chi' + a\cos\chi, \qquad \frac{a\,d\theta}{C'Q} = k\frac{d\chi}{\cos\chi'},$$

$$(8) \quad G = \frac{4nk}{a}\int_0^\pi \sin^2(\chi+\chi')\frac{d\chi}{\cos\chi'} = \frac{8n}{3\,OC'}\,\frac{(1+k^2)H-(1-k^2)K}{k^2},$$

and G is given by the same expression when C' is moved to any point on the axis of the tore, the modular angle β of K and H being half the angle between the tangents from C' to the circular section of the tore.

33. The form of the result shows that the integral for G and dM/db is the same essentially; for the comparison of these and other integrals it is convenient at this stage to have the following integrals to quote in the sequel:

$$(1) \qquad \int_0^{2\pi}\frac{a\,d\theta}{PQ} = \frac{4aF}{r_1} = \frac{8aK}{r_1+r_2},$$

the potential of the ring AB; and as in § 31,

$$(2) \quad \int\frac{2aA(1+\cos\theta)d\theta}{PQ^3} = \frac{4}{r_1}\int_{t_3}^{t_2}\frac{t_2-t}{t_1-t}\frac{\sqrt{(t_1-t_3)}\,dt}{\sqrt{T}} = \frac{4}{r_1}(F-E) = 4A\frac{G_1}{r_1}$$

$$(3) \quad \int\frac{2aA(1-\cos\theta)d\theta}{PQ^3} = \frac{4}{r_1}\int\frac{t-t_3}{t_1-t}\frac{\sqrt{(t_1-t_3)}\,dt}{\sqrt{T}} = \frac{4}{r_1}\,\frac{E-c'^2F}{c'^2} = 4A\frac{G_2}{r_2}.$$

Since

$$\frac{d}{d\theta}\frac{2aA\sin\theta}{PQ} = \frac{2aA\cos\theta}{PQ} + \frac{2a^2A^2\sin^2\theta}{PQ^3}$$

$$(4) \qquad = \frac{PQ^2-\tfrac12(r_1^2+r_2^2)}{PQ} + \frac{(r_1^2-PQ^2)(PQ^2-r_2^2)}{2PQ^3}$$

$$= \tfrac12 PQ - \tfrac12\frac{r_1^2r_2^2}{PQ^3},$$

$$(5) \qquad \int_0^{2\pi}PQ\,d\theta = \int\frac{r_1^2r_2^2}{PQ^3}\,d\theta = 4r_1E = 2(r_1+r_2)(2H-k'^2K),$$

$$(6) \qquad \int\frac{\cos\theta\,d\theta}{PQ} = \frac{4}{r_1}\,\frac{2E-(1+c'^2)F}{c^2} = -\frac{8}{r_1+r_2}\,\frac{K-H}{k}.$$

Also

$$\frac{d}{d\theta}(2aA\sin\theta PQ) = \left(PQ^2-\frac{r_1^2+r_2^2}{2}\right)PQ - \frac{(r_1^2-PQ^2)(PQ^2-r_2^2)}{2PQ}$$

$$(7) \qquad = \tfrac32 PQ^3 - (r_1^2+r_2^2)PQ + \frac{r_1^2r_2^2}{2PQ},$$

$$3\int_0^{2\pi} PQ^3 d\theta = 2(r_1^2 + r_2^2)\int PQ d\theta - r_1^2 r_2^2 \int \frac{d\theta}{PQ}$$

$$(8) \qquad = 8r_1(r_1^2 + r_2^2)E - 4r_1 r_2^2 F$$

$$= 4r_1^3 [2(1 + c'^2)E - c'^2 F]$$

$$= (r_1 + r_2)^3 [4(1 + k^2)H - \tfrac{1}{2}k'^2 (5 + 3k^2)K],$$

$$(9) \qquad 3\int \frac{\sin^2\theta d\theta}{PQ} = \frac{16}{r_1}\frac{(1+c'^2)E-2c'^2F}{c^4} = \frac{8}{r_1+r_2}\frac{(1+k^2)H-(1-k_2^2)K}{k^2},$$

$$(10) \quad 3\int PQ\cos\theta d\theta = 4r_1\frac{(1+c'^2)E-2c'^2F}{c^2} = 2(r_1+r_2)\frac{(1+k^2)H-(1-k^2)K}{k},$$

so that as is evident from (7) these last two integrals (9) and (10) are essentially the same as well as the integral (1) § 30; and this shows that G is also the potential of the tore at any point C' of its axis, the tore being homogeneous and of mass $2\pi n$.

Potential of an elliptic disc, gravitational and magnetic.

34. Begin now with the potential V of an elliptic disc

$$(1) \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{0} = 1$$

of uniform surface density 1; then (Cayley, Proceedings London Mathematical Society, vol. 6, p. 42)

$$(2) \qquad V = 2ab\int_\lambda^\infty \sqrt{\left(1 - \frac{x^2}{a^2+\epsilon} - \frac{y^2}{b^2+\epsilon} - \frac{z^2}{\epsilon}\right)}\frac{d\epsilon}{\sqrt{(a^2+\epsilon\cdot b^2+\epsilon\cdot\epsilon)}},$$

where λ is the positive root of

$$(3) \qquad 1 - \frac{x^2}{a^2+\epsilon} - \frac{y^2}{b^2+\epsilon} - \frac{z^2}{\epsilon} = 0,$$

μ and ν denoting the other two.

Next by differentiation of V with respect to z the magnetic potential Ω is obtained of the elliptic disc, magnetized normally with uniform unit intensity, as represented by the component attraction of the disc perpendicular to the plane, or by the solid or conical angle subtended at a point; so that

$$(4) \qquad \Omega = -\frac{dV}{dz} = \int_\lambda^\infty \frac{2abz d\epsilon}{\epsilon\sqrt{\left(1 - \frac{x^2}{a^2+\epsilon} - \frac{y^2}{b^2+\epsilon} - \frac{z^2}{\epsilon}\right)}\sqrt{(a^2+\epsilon\cdot b^2+\epsilon\cdot\epsilon)}},$$

reducing to

$$(5)\qquad \Omega = \int_\lambda^\infty \frac{4abz\,d\epsilon}{\epsilon\sqrt{(4\cdot\epsilon - \lambda\cdot\epsilon - \mu\cdot\epsilon - \nu)}} = \int_\lambda^\infty \frac{4\sqrt{(\lambda\mu\nu)}}{\epsilon}\,\frac{d\epsilon}{\sqrt{E}},$$

$$E = 4\cdot\epsilon - \lambda\cdot\epsilon - \mu\cdot\epsilon - \nu.$$

This is proved by determining x, y, z in terms of λ, μ, ν, the elliptic coördinates, such that (x, y, z) is a point of intersection of the three confocals of (3) to the elliptic disc given in (1), when

$$(6)\qquad \epsilon = \lambda,\ \mu,\ \nu,$$

$$(7)\qquad \infty > \lambda > 0 > \mu > -b^2 > \nu > -a^2 > -\infty,$$

and then

$$(8)\ x^2 = \frac{a^2+\lambda\cdot a^2+\mu\cdot a^2+\nu}{a^2-b^2\cdot a^2-0},\quad y^2 = \frac{b^2+\lambda\cdot b^2+\mu\cdot b^2+\nu}{b^2-a^2\cdot b^2-0},\quad z^2 = \frac{0+\lambda\cdot 0+\mu\cdot 0+\nu}{a^2-0\cdot b^2-0},$$

$$(9)\qquad 1 - \frac{x^2}{a^2+\epsilon} - \frac{y^2}{b^2+\epsilon} - \frac{z^2}{\epsilon} = \frac{\epsilon-\lambda\cdot\epsilon-\mu\cdot\epsilon-\nu}{a^2+\epsilon\cdot b^2+\epsilon\cdot\epsilon}.$$

Interpreted geometrically, Ω is the solid or conical angle subtended by the elliptic disc (1), or the *apparent area*, as it may be called; and SCHWARZ has shown that the apparent area or *Scheinbare Grösse* of the ellipsoid or of any elliptic section of the tangent cone is given by an expression equally simple (Göttinger Nachrichten, 1885),

$$(10)\ \Omega = \int_\lambda^\infty \frac{2\sqrt{(-4\cdot\lambda_0-\lambda\cdot\lambda_0-\mu\cdot\lambda_0-\nu)}\,d\epsilon}{(\epsilon-\lambda_0)\sqrt{(4\cdot\epsilon-\lambda\cdot\epsilon-\mu\cdot\epsilon-\nu)}} = 2\pi - \int_\nu^\mu \frac{2\sqrt{(-E_0)}}{\lambda_0-\epsilon}\,\frac{d\epsilon}{\sqrt{E}}$$

$$(E_0 = 4\cdot\lambda_0 - \lambda\cdot\lambda_0 - \mu\cdot\lambda_0 - \nu),$$

where the ellipsoid is defined by

$$(11)\qquad \frac{x^2}{a^2+\lambda_0} + \frac{y^2}{b^2+\lambda_0} + \frac{z^2}{\lambda_0} = 1,$$

reducing to the elliptic disc as its focal ellipse when $\lambda_0 = 0$.

Put

$$(12)\quad \epsilon-\lambda = M^2(s-s_1),\qquad \epsilon-\mu = M^2(s-s_2),\qquad \epsilon-\nu = M^2(s-s_3)$$

and

$$\epsilon(\text{or } \epsilon-\lambda_0) = M^2(s-\sigma),$$

$$(13)\qquad E = 4\cdot\epsilon - \lambda\cdot\epsilon - \mu\cdot\epsilon - \nu = M^6 S,$$

$$(14)\qquad 4a^2b^2z^2 = 4\lambda\mu\nu = -E_0 = -M^6\Sigma;$$

and then, as in (3), § 5,

$$(15)\qquad \Omega = \int_{s_1}^\infty \frac{2\sqrt{-\Sigma}}{s-\sigma}\,\frac{ds}{\sqrt{S}} = 4A = 2\pi - 4B.$$

35. SCHWARZ's theorem is proved by considering the equation of the tangent cone from (x, y, z) to the ellipsoid defined by $\epsilon = \lambda_0$, which, when referred to its principal axes, the normals of the three confocals through (x, y, z) defined by $\epsilon = \lambda, \mu, \nu$ becomes (SALMON, *Solid Geometry*, § 173),

$$(1) \qquad \frac{x^2}{\lambda - \lambda_0} + \frac{y^2}{\mu - \lambda_0} + \frac{z^2}{\nu - \lambda_0} = 0.$$

one of a family of cones confocal for different values of λ_0.

The conical angle Ω of this tangent cone, and the perimeter Φ of the sphero-conic on unit sphere of the reciprocal cone

$$(2) \qquad (\lambda - \lambda_0)x^2 + (\mu - \lambda_0)y^2 + (\nu - \lambda_0)z^2 = 0$$

are connected by the relation

$$(3) \qquad \Omega + \Phi = 2\pi.$$

On the reciprocal cone replacing x, y, z by direction cosines l, m, n, we may take

$$(4) \qquad l^2 = k(\mu - \nu)\left(\frac{1}{\lambda - \lambda_0} - \frac{1}{\epsilon - \lambda_0}\right),$$

$$m^2 = k(\nu - \lambda)\left(\frac{1}{\mu - \lambda_0} - \frac{1}{\epsilon - \lambda_0}\right),$$

$$n^2 = k(\lambda - \mu)\left(\frac{1}{\nu - \lambda_0} - \frac{1}{\epsilon - \lambda_0}\right),$$

subject to the condition

$$(5) \qquad l^2 + m^2 + n^2 = k\left(\frac{\mu - \nu}{\lambda - \lambda_0} + \frac{\nu - \lambda}{\mu - \mu_0} + \frac{\lambda - \mu}{\nu - \nu_0}\right) = 1,$$

$$(6) \qquad k = -\frac{\lambda - \lambda_0 \cdot \mu - \lambda_0 \cdot \nu - \lambda_0}{\mu - \nu \cdot \nu - \lambda \cdot \lambda - \mu};$$

and now

$$(7) \qquad (d\Phi)^2 = dl^2 + dm^2 + dn^2 = \frac{-E_0}{4(\epsilon - \lambda_0)^2}\frac{d\epsilon^2}{E},$$

$$(8) \qquad E = 4 \cdot \epsilon - \lambda \cdot \epsilon - \mu \cdot \epsilon - \nu, \qquad E_0 = 4 \cdot \lambda_0 - \lambda \cdot \lambda_0 - \mu \cdot \lambda_0 - \nu.$$

As a point travels round a quadrant of the sphero-conic, ϵ decreases from μ to ν, and

$$(9) \qquad \lambda > \lambda_0 > 0 > \mu > \epsilon > \nu,$$

$$(10) \qquad \Phi = 4\int_\nu^\mu \frac{\frac{1}{2}\sqrt{(-E_0)}}{\lambda_0 - \epsilon}\frac{d\epsilon}{\sqrt{E}},$$

$$(11) \qquad \Omega = 2\pi - \Phi = 4\int_\lambda^\infty \frac{\frac{1}{2}\sqrt{(-E_0)}}{\epsilon - \lambda_0}\frac{d\epsilon}{\sqrt{E}}.$$

in accordance with (5) § 5, Ω and Φ being $4A$ and $4B$, SCHWARZ's result.

36. In the notation of GAUSS (CAYLEY, Proceedings of the London Mathematical Society, vol. 6; HALPHEN, *Fonctions Elliptiques*, vol. 2, chap. 8)

$$G,\ G'',\ G' = \lambda,\ -\mu,\ -\nu, \tag{1}$$

and in the region $\mu > \epsilon > \nu$ we can put

$$G + G' \cos^2 T + G'' \sin^2 T = \lambda - \epsilon, \tag{2}$$

$$G' \cos^2 T + G'' \sin^2 T = \lambda_0 - \epsilon, \tag{3}$$

$$(G' - G'') \cos^2 T = \mu - \epsilon,\ (G' - G'') \sin^2 T = \epsilon - \nu, \tag{4}$$

$$dT = \frac{d\epsilon}{\sqrt{(4 \cdot \mu - \epsilon \cdot \epsilon - \nu)}}, \tag{5}$$

$$U = \int_0^T \frac{dT}{\sqrt{(G + G' \cos^2 T + G'' \sin^2 T)}} = \int_{-\nu}^{\epsilon} \frac{d\epsilon}{\sqrt{E}}, \tag{6}$$

$$U\sqrt{(\lambda - \nu)} = hK, \qquad T = \operatorname{am} hK. \tag{7}$$

In the region $\infty > \epsilon > \lambda > \lambda_0$, we should take

$$\sin^2 T = \frac{\lambda - \nu}{\epsilon - \nu}, \qquad \cos^2 T = \frac{\epsilon - \lambda}{\epsilon - \nu}, \qquad T = \operatorname{am} hK, \tag{8}$$

$$G + G' \cos^2 T + G'' \sin^2 T = \frac{\lambda - \nu \cdot \epsilon - \mu}{\epsilon - \nu}, \tag{9}$$

$$G' \cos^2 T + G'' \sin^2 T = \frac{(\lambda_0 - \nu \cdot \epsilon - \nu) - (\lambda - \nu \cdot \mu - \nu)}{\epsilon - \nu}, \tag{10}$$

equivalent to the substitution

$$\left(\epsilon - \nu,\ \frac{\lambda - \nu \cdot \mu - \nu}{\epsilon - \nu}\right). \tag{11}$$

37. The inequalities

$$\infty > \epsilon > \lambda > \lambda_0 > 0 > \mu > -b^2 > \nu > -a^2, \tag{1}$$

$$\infty > s > s_1 > \sigma > s_2 > s > s_3 > -\infty, \tag{2}$$

show that the form $A(fK')$ or $B(fK')$ is required; and from (3) § 5,

$$\Omega = 2\pi(1 - f) - 4K \operatorname{zn} fK', \tag{3}$$

equivalent to NAGAOKA's result for a circle expressed by Weierstrass functions; and here, as in (6), (7), (8), (9) § 5,

(4) $$\operatorname{zn} fK' = ZfK' = E\phi - fE,$$

(5) $$fK' = F(\phi, k'), \qquad \phi = \operatorname{am} fK',$$

so that LEGENDRE's Table IX can be employed for a numerical application,

(6) $$\sin^2\phi = \operatorname{sn}^2 fK' = \frac{s_1 - \sigma}{s_1 - s_2} = \frac{\lambda}{\lambda - \mu},$$

(7) $$\cos^2\phi = \operatorname{cn}^2 fK' = \frac{\sigma - s_2}{s_1 - s_2} = \frac{-\mu}{\lambda - \mu},$$

(8) $$\Delta^2\phi = \operatorname{dn}^2 fK' = \frac{\sigma - s_3}{s_1 - s_3} = \frac{-\nu}{\lambda - \nu},$$

(9) $$k^2 = \frac{s_2 - s_3}{s_1 - s_3} = \frac{\mu - \nu}{\lambda - \nu}, \qquad k'^2 = \frac{s_1 - s_2}{s_1 - s_3} = \frac{\lambda - \mu}{\lambda - \nu},$$

and the modular angle is then half the angle between the generating lines of the confocal hyperboloid of one sheet, or the focal lines of the tangent cone.

38. Thus for instance, if we take $f = \frac{1}{2}$,

(1) $$\Delta^2\phi = \frac{-\nu}{\lambda - \nu} = k = \sqrt{\frac{\mu - \nu}{\lambda - \nu}}, \qquad \frac{1}{\lambda} + \frac{1}{\mu} - \frac{1}{\nu} = 0, \qquad k = \frac{-\mu}{\lambda},$$
$$\mu = -k\lambda, \qquad \nu = -\frac{k\lambda}{1 - k},$$

and now

(2) $$\operatorname{zn} \tfrac{1}{2}K' = \tfrac{1}{2}(1 - k), \qquad \Omega = \pi - 2(1 - k)K.$$

At an infinite distance,

(3) $$\lambda = \infty, \quad \sin\phi = 1, \quad k = 0, \quad K = \tfrac{1}{2}\pi, \quad k' = 1, \quad K' = \infty, \quad E' = 1,$$

(4) $$\operatorname{zn} fK' = E\phi - fE' = \sin\phi - f = 1 - f,$$

so that

(5) $$\Omega = 2\pi(1 - f) - 2\pi(1 - f) = 0,$$

a verification.

Inside the focal ellipse, in its plane,

(6) $$\lambda = 0, \qquad f = 0, \qquad \Omega = 2\pi;$$

but in the plane outside the ellipse

(7) $$\mu = 0, \qquad f = 1, \qquad \operatorname{zn} K' = 0, \qquad \Omega = 0,$$

to serve as a verification.

Along the edge of the focal ellipse

$$(8) \qquad \lambda = 0, \qquad \mu = 0, \qquad k' = 0, \qquad \mathrm{zn}\, fK' = 0,$$

$$(9) \qquad \Omega = 2\pi(1 - f),$$

and two surfaces of constant Ω_1 and Ω_2 intersect along the edge at an angle

$$(10) \qquad \tfrac{1}{2}(\Omega_2 - \Omega_1) = \pi(f_1 - f_2);$$

see MAXWELL, E. and M., § 487.

Along the focal hyperbola

$$(11) \qquad \mu = \nu = -b^2, \qquad k = 0, \qquad K = \tfrac{1}{2}\pi,$$

$$k' = 1, \qquad E' = 1, \qquad E\phi = \sin\phi,$$

$$\mathrm{zn}\, fK' = \sin\phi - f,$$

so that

$$(12) \qquad \Omega = 2\pi(1 - \sin\phi) = 2\pi\left(1 - \sqrt{\frac{\lambda}{\lambda - \mu}}\right),$$

and the conical angle is now bounded by a cone of revolution of vertical angle $\pi - 2\phi$ (CAYLEY, Proceedings of the London Mathematical Society, vol. 6).

On the axis perpendicular to the plane of the ellipse

$$(13) \qquad x = 0, \qquad y = 0, \qquad z^2 = \lambda, \qquad b^2 + \mu = 0, \qquad a^2 + \nu = 0,$$

$$k^2 = \frac{a^2 - b^2}{a^2 + z^2}, \qquad k'^2 = \frac{b^2 + z^2}{a^2 + z^2},$$

$$\mathrm{sn}^2 fK' = \frac{z^2}{b^2 + z^2}, \qquad \mathrm{cn}^2 fK' = \frac{b^2}{b^2 + z^2}, \qquad \mathrm{dn}^2 fK' = \frac{a^2}{a^2 + z^2},$$

$$\mathrm{dn}^2 (1 - f) K' = 1 - \frac{b^2}{a^2} = e^2.$$

39. The equipotential surface $\Omega = 2\pi f$, a constant, cuts the plane of the ellipse (or any other area whatever) along its edge at a constant angle πf; this is evident from simple geometrical considerations of the conical angle or apparent area of the ellipse as seen from a point close to the edge, when the apparent area on the unit sphere is cut out by two diametral planes, one parallel to the plane of the area, and the other drawn through the adjacent tangent of the area.

Magnetic interpretation of the potential of the elliptic disc.

40. In the potential of an elliptic disc, as given by CAYLEY,

$$(1) \qquad V = 4ab \int_\lambda^\infty \left(1 - \frac{z^2}{\epsilon} - \frac{y^2}{b^2 + \epsilon} - \frac{x^2}{a^2 + \epsilon}\right) \frac{d\epsilon}{\sqrt{E}},$$

when modified by (9) § 34, and employing (12) § 34, with

$$(2)\quad \epsilon = m^2(s-\sigma), \qquad b^2+\epsilon = m^2(s-\sigma'), \qquad a^2+\epsilon = m^2(s-\sigma''),$$

$$(3)\quad \infty > s > s_1 > \sigma > s_2 > \sigma' > s_3 > \sigma'' > -\infty,$$

$$\lambda = m^2(s_1-\sigma), \qquad b^2+\lambda = m^2(s_1-\sigma'), \qquad a^2+\lambda = m^2(s_1-\sigma''),$$

$$(4)\quad \mu = -m^2(\sigma-s_2), \qquad b^2+\mu = -m^2(s_2-\sigma'), \qquad a^2+\mu = m^2(s_2-\sigma''),$$

$$\nu = -m^2(\sigma-s_1), \qquad b^2+\nu = -m^2(\sigma'-s_3), \qquad a^2+\nu = m^2(s_3-\sigma''),$$

$$(5)\quad a^2b^2z^2 = \lambda\mu\nu = -\tfrac{1}{4}m^6\Sigma,$$

$$(6)\quad (a^2-b^2)b^2y^2 = b^2+\lambda\cdot b^2+\mu\cdot b^2+\nu = \tfrac{1}{4}m^6\Sigma',$$

$$(7)\quad (a^2-b^2)a^2x^2 = a^2+\lambda\cdot a^2+\mu\cdot a^2+\nu = -\tfrac{1}{4}m^6\Sigma'',$$

and the terms in V are

$$(8)\quad 4ab\int_\lambda^\infty \frac{d\epsilon}{\sqrt{E}} = \frac{4abK}{\sqrt{(\lambda-\nu)}},$$

$$(9)\quad 4abz^2\int \frac{d\epsilon}{\epsilon\sqrt{E}} = 4z\int_{s_1}^\infty \frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s-\sigma}\,\frac{ds}{\sqrt{S}}$$

$$= 4zA(fK') = 4z\left[\tfrac{1}{2}\pi(1-f) - K\,\mathrm{zn}\,fK'\right],$$

$$(10)\quad 4aby^2\int \frac{d\epsilon}{(b^2+\epsilon)\sqrt{E}} = \frac{4ay}{\sqrt{(a^2-b^2)}}\int \frac{\frac{1}{2}\sqrt{(\Sigma')}}{s-\sigma'}\,\frac{ds}{\sqrt{S}}$$

$$= \frac{4ay}{\sqrt{(a^2-b^2)}}\,E(f'K) = \frac{4ay}{\sqrt{(a^2-b^2)}}\,K\,\mathrm{zn}\,f'K,$$

$$(11)\quad 4abx^2\int \frac{d\epsilon}{(a^2+\epsilon)\sqrt{E}} = \frac{4bx}{\sqrt{(a^2-b^2)}}\int \frac{\frac{1}{2}\sqrt{(-\Sigma'')}}{s-\sigma''}\,\frac{ds}{\sqrt{S}}$$

$$= \frac{4bx}{\sqrt{(a^2-b^2)}}\,C(f''K')$$

$$= \frac{4bx}{\sqrt{(a^2-b^2)}}\left[K\,\mathrm{zs}\,f''K' - \tfrac{1}{2}\pi(1-f'')\right].$$

Here, as in (3) § 2,

$$(12)\quad f = \int_\sigma^{s_1} \div \int_{s_2}^{s_1} \frac{d\sigma}{\sqrt{(-\Sigma)}}, \quad \mathrm{sn}^2 fK' = \frac{\lambda}{\lambda-\mu}, \quad \mathrm{cn}^2 fK' = \frac{-\mu}{\lambda-\mu}, \quad \mathrm{dn}^2 fK' = \frac{-\nu}{\lambda-\nu}$$

$$(13)\quad f' = \int_{s_3}^{\sigma'} \div \int_{s_3}^{s_2} \frac{ds}{\sqrt{S}}, \quad \mathrm{sn}^2 f'K = \frac{-b^2-\nu}{\mu-\nu}, \quad \mathrm{cn}^2 f'K = \frac{b^2+\mu}{\mu-\nu}, \quad \mathrm{dn}^2 f'K = \frac{b^2+\lambda}{\lambda-\nu}$$

$$(14)\quad f'' = \int_{-\infty}^{\sigma''} \div \int_{-\infty}^{s_3} \frac{d\sigma}{\sqrt{(-\Sigma)}}, \quad \mathrm{sn}^2 f''K' = \frac{\lambda-\nu}{a^2+\lambda}, \quad \mathrm{cn}^2 f''K' = \frac{a^2+\nu}{a^2+\lambda}, \quad \mathrm{dn}^2 f''K' = \frac{a^2+\mu}{a^2+\lambda}$$

We have seen already in (5) and (15) § 34, that

$$(15)\quad \Omega = -\frac{dV}{dz} = 4A(fK') = 2\pi(1-f) - 4K\,\mathrm{zn}\,fK',$$

is the magnetic potential of the elliptic disc with uniform normal magnetization parallel to Oz, or of the unit current round the elliptic ring; so also

$$(16)\quad \Omega' = -\frac{dV}{dy} = \frac{4a}{\sqrt{(a^2-b^2)}}E(f'K) = \frac{4a}{\sqrt{(a^2-b^2)}}K\,\mathrm{zn}\,f'K,$$

is the potential for uniform magnetization parallel to Oy the minor axis; and

$$(17)\quad \Omega'' = -\frac{dV}{dx} = \frac{4b}{\sqrt{(a^2-b^2)}}C(f''K') = \frac{4b}{\sqrt{(a^2-b^2)}}[K\,\mathrm{zs}\,f''K' - \tfrac{1}{2}\pi(1-f'')],$$

for uniform magnetization parallel to Ox the major axis: and Ω', Ω'' are the components F, G of the vector potential of the elliptic current.

For uniform magnetization of the elliptic disc with direction cosines l, m, n, the magnetic potential is

$$(18)\quad n\Omega + m\Omega' + l\Omega''.$$

Induced magnetism of a hollow ellipsoid of soft iron.

41. We may cite here the expression of the induction of a uniform magnetic field on a hollow ellipsoid of soft iron, bounded by confocals, of the family

$$(1)\quad \frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{\lambda} = 1,$$

λ_0 and λ_1 defining the outer and inner surface of the shell.

We require the three incomplete II. E. I.'s

$$(2)\quad A, B, C = \int_\lambda^\infty \frac{d\epsilon}{(a^2+\epsilon, b^2+\epsilon, \epsilon)\sqrt{P}},$$

$$P = 4\cdot a^2+\epsilon\cdot b^2+\epsilon\cdot\epsilon, \qquad A+B+C = \frac{1}{\sqrt{(a^2+\lambda\cdot b^2+\lambda\cdot\lambda)}};$$

and with the fraction h defined as in § 2 by

$$(3)\quad h = \int_\lambda^\infty \div \int_0^\infty \frac{d\epsilon}{\sqrt{P}},$$

we find from § 3,

$$(4) \qquad A = \frac{1}{a(a^2-b^2)}\left[h(K-H) - \operatorname{zn} hK\right],$$

$$(5) \qquad B = \frac{a}{b^2(a^2-b^2)}\left[h(H-k'^2K) - \operatorname{zn}(1-h)K\right],$$

$$(6) \qquad C = \frac{1}{ab^2}\left[-hH + \operatorname{zs}(1-h)K\right],$$

$$(7) \qquad k^2 = 1 - \frac{b^2}{a^2}, \qquad k' = \frac{b}{a}.$$

If the external inducing magnetic field has a potential

$$(8) \qquad V = Xx + Yy + Zz,$$

we can examine the components one at a time, so that if the component Xx induces a magnetic potential Ω in the iron of the shell, Ω_0 in the outside space, and Ω_1 in the cavity, the magnetic conditions are satisfied by putting

$$(9) \qquad \Omega_1 = L_1 x, \qquad \Omega_0 = L_0 \frac{A}{A_0} x,$$

$$(10) \qquad \Omega = L_0 \frac{A_1 - A}{A_1 - A_0} x + L_1 \frac{A - A_0}{A_1 - A_0} x,$$

where

$$(11) \qquad \frac{A_1 - A}{A_1 - A_0} = \frac{(h_1 - h)(K-H) - (\operatorname{zn} h_1 K - \operatorname{zn} hK)}{(h_1 - h_0)(K-H) - (\operatorname{zn} h_1 K - \operatorname{zn} h_0 K)}, \cdots;$$

and then determining L_0 and L_1 from the condition, with magnetic permeability μ,

$$(12) \qquad \mu \frac{d}{d\nu}(Xx + \Omega) = \frac{d}{d\nu_1}(Xx + \Omega_1) \text{ or } \frac{d}{d\nu_0}(Xx + \Omega_0)$$

in crossing the inner or outer surface, noticing that

$$(13) \qquad x\frac{dA}{d\nu} = -\frac{x}{a^2+\lambda}\frac{1}{\sqrt{P}}\frac{d\lambda}{d\nu} = -\frac{2px}{a^2+\lambda}\frac{1}{\sqrt{P}} = -\frac{2l}{\sqrt{P}}.$$

Other cases for components of V such as xy, xyz, $\cdots$ are given in the Journal de Physique (1881); the hydrodynamical analogues can be developed by means of the same analysis.

Potential of a circular disc, gravitational and magnetic.

42. For a circular disc put $a^2 = b^2$,

$$(1)\qquad V = 2a^2 \int_\lambda^\infty \sqrt{\left(1 - \frac{x^2+y^2}{a^2+\epsilon} - \frac{z^2}{\epsilon}\right)} \frac{d\epsilon}{(a^2+\epsilon)\sqrt{\epsilon}};$$

ellipsoidal coördinates are now unsuitable, and the elliptic coördinates, v and w, must be employed, with the substitution

$$(2)\qquad a^2 + \epsilon = a^2 \operatorname{ch}^2 u, \qquad \epsilon = a^2 \operatorname{sh}^2 u,$$

$$(3)\qquad x^2 + y^2 = a^2 \operatorname{ch}^2 v \cos^2 w, \qquad z^2 = a^2 \operatorname{sh}^2 v \sin^2 w,$$

$$(4)\qquad r_1,\ r_2 = a(\operatorname{ch} v \pm \cos w),$$

r_1, r_2 denoting the focal distances of P from the foci A, B in fig. A; and

$$(5)\qquad \begin{aligned} 1 - \frac{x^2+y^2}{a^2+\epsilon} - \frac{z^2}{\epsilon} &= 1 - \frac{\operatorname{ch}^2 v \cos^2 w}{\operatorname{ch}^2 u} - \frac{\operatorname{sh}^2 v \sin^2 w}{\operatorname{sh}^2 u} \\ &= \frac{(\operatorname{ch}^2 u - \operatorname{ch}^2 v)(\operatorname{ch}^2 u - \cos^2 w)}{\operatorname{ch}^2 u \operatorname{sh}^2 u}, \end{aligned}$$

$$(6)\qquad V = 4a \int_v^\infty \frac{\sqrt{(\operatorname{ch}^2 u - \operatorname{ch}^2 v \cdot \operatorname{ch}^2 u - \cos^2 w)}}{\operatorname{ch}^2 u \operatorname{sh} u} du,$$

$$(7)\qquad \begin{aligned} \Omega = -\frac{dV}{dz} &= \int_\lambda^\infty \frac{2a^2 z d\epsilon}{\epsilon(a^2+\epsilon)\sqrt{\epsilon}\sqrt{\left(1 - \frac{x^2+y^2}{a^2+\epsilon} - \frac{z^2}{\epsilon}\right)}} \\ &= 4\int_v^\infty \frac{\operatorname{sh} v \sin w du}{\operatorname{sh} u \sqrt{(\operatorname{ch}^2 u - \operatorname{ch}^2 v \cdot \operatorname{ch}^2 u - \cos^2 w)}}. \end{aligned}$$

Next substitute

$$(8)\qquad \operatorname{sh}^2 u = m^2(s - \sigma), \qquad \operatorname{ch}^2 u = m^2(s - s_3),$$

$$(9)\qquad m^2(\sigma - s_3) = 1, \qquad \operatorname{ch}^2 u = \frac{s - s_3}{\sigma - s_3}, \qquad \operatorname{sh}^2 u = \frac{s - \sigma}{\sigma - s_3},$$

$$(10)\qquad \operatorname{ch}^2 u - \operatorname{ch}^2 v = \frac{s - s_1}{\sigma - s_3}, \qquad \operatorname{ch}^2 u - \cos^2 w = \frac{s - s_2}{\sigma - s_3},$$

$$(11)\qquad \operatorname{ch}^2 v = \frac{s_1 - s_3}{\sigma - s_3}, \qquad \operatorname{sh}^2 v = \frac{s_1 - \sigma}{\sigma - s_3},$$

$$(12)\qquad \cos^2 w = \frac{s_2 - s_3}{\sigma - s_3}, \qquad \sin^2 w = \frac{\sigma - s_2}{\sigma - s_3},$$

$$(13)\qquad k^2 = \frac{s_2 - s_3}{s_1 - s_3} = \frac{\cos^2 w}{\operatorname{ch}^2 v}, \qquad k = \frac{r_1 - r_2}{r_1 + r_2},$$

as in Maxwell's second expression for M, § 701.

The inequality sequence runs

$$(14)\qquad \infty > s > s_1 > \sigma > s_2 > s_3 > -\infty,$$

so that, from § 2, dropping K',

$$(15)\qquad \operatorname{ch} v = \frac{1}{\operatorname{dn} f} = \frac{\operatorname{dn}(1-f)}{k}, \qquad \cos w = \operatorname{dn}(1-f).$$

If then we write A for $\surd(x^2+y^2)$, $x = A\cos\phi$, $y = A\sin\phi$,

$$(16)\qquad \frac{A}{a} = \operatorname{ch} v \cos w = \frac{\operatorname{dn}(1-f)}{\operatorname{dn} f} = \frac{k}{\operatorname{dn}^2 f} = \frac{\operatorname{dn}^2(1-f)}{k},$$

$$(17)\qquad \frac{z}{a} = \operatorname{sh} v \operatorname{sn} w = \frac{k'^2 \operatorname{sn} f \operatorname{cn} f}{\operatorname{dn}^2 f} = \frac{k'^2}{k}\operatorname{sn}(1-f)\operatorname{cn}(1-f),$$

$$(18)\qquad \frac{r_1,\, r_2}{a} = \operatorname{ch} v \pm \cos w = \left(\frac{1}{k} \pm 1\right)\operatorname{dn}(1-f).$$

Since

$$(19)\qquad \frac{\operatorname{sh} u\,du}{\surd(\operatorname{ch}^2 u - \operatorname{ch}^2 v \cdot \operatorname{ch}^2 u - \cos^2 w)} = \frac{\surd(\sigma - s_3)\,ds}{\surd S},$$

equation (7), as before in (15) § 34, becomes transformed to

$$(20)\qquad \Omega = 4\int_{s_1}^{\infty} \frac{\tfrac{1}{2}\surd(-\Sigma)}{s-\sigma}\,\frac{ds}{\surd S} = 4A(fK') = 2\pi(1-f) - 4K \operatorname{zn} fK'.$$

To make s oscillate between s_2 and s_3, substitute

$$(21)\qquad \operatorname{ch}^2 u = \frac{\sigma - s_3}{s - s_3}, \quad \operatorname{sh}^2 u = \frac{\sigma - s}{s - s_3}, \quad \operatorname{ch}^2 v = \frac{\sigma - s_3}{s_2 - s_3}, \quad \cos^2 w = \frac{\sigma - s_3}{s_2 - s_3}.$$

43. The potential for uniform magnetization of the circular disc parallel to AB is, as before in (16), (17), § 40,

$$(1)\qquad \Omega' = -\frac{dV}{dA} = \int_{\lambda}^{\infty} \frac{2a^2 A\,d\epsilon}{(a^2+\epsilon)^2\surd\epsilon\sqrt{\left(1 - \dfrac{A^2}{a^2+\epsilon} - \dfrac{z^2}{\epsilon}\right)}}$$

$$= \int_{v}^{\infty} \frac{4\operatorname{ch} v \cos w \operatorname{sh} u \operatorname{dn}}{\operatorname{ch}^2 u\,\surd(\operatorname{ch}^2 u - \operatorname{ch}^2 v \cdot \operatorname{ch}^2 u - \cos^2 w)}$$

$$= \frac{4}{\cos w}\int_{s_1}^{\infty} \frac{s_2 - s_3}{s - s_3}\,\frac{\surd(s_1 - s_3)\,ds}{\surd S} = 4\,\frac{K - H}{\cos w}.$$

From MAXWELL's second expression in § 701,

$$(2)\qquad M = 4\pi(r_1 + r_2)(K - H) = 8\pi a \operatorname{ch} v (K - H) = 2\pi A \Omega',$$

and this is evident when we notice in figure B that M is composed of elements

$$(3)\qquad \frac{\cos\theta}{PQ''} a d\theta - \frac{\cos\theta}{PQ} a d\theta,$$

which is the magnetic potential at P of a strip QQ'' parallel to AB, magnetized longitudinally; and these elements are integrated over the circle AQB, and multiplied afterwards by $2\pi A$ to obtain M for the two circles.

Replace A by x, then at any point $(x, 0, z)$

$$(4)\qquad M = 2\pi x \Omega' = -2\pi x \frac{dV}{dx}, \qquad 2\pi\Omega = -2\pi \frac{dV}{dz},$$

$$(5)\qquad 2\pi \frac{d\Omega}{dx} = -2\pi \frac{d^2 V}{dx\,dz} = \frac{1}{x}\frac{dM}{dz},$$

$$(6)\qquad 2\pi \frac{d\Omega}{dz} = -2\pi \frac{d^2 V}{dz^2} = 2\pi\left(\frac{d^2 V}{dx^2} + \frac{1}{x}\frac{dV}{dx}\right) = -\frac{1}{x}\frac{dM}{dx},$$

and a magnetic line of force along which M is constant is the orthogonal trajectory of the equipotential surfaces Ω = constant; in fact M is the Stokes function of the magnetic potential, and

$$(7)\qquad \frac{d^2\Omega}{dz^2} + \frac{d^2\Omega}{dx^2} + \frac{1}{x}\frac{d\Omega}{dx} = 0,$$

$$(8)\qquad \frac{d^2 M}{dz^2} + \frac{d^2 M}{dx^2} - \frac{1}{x}\frac{dM}{dx} = 0,$$

or as they may be written

$$(9)\qquad \frac{d}{dz}\left(x\frac{d\Omega}{dz}\right) + \frac{d}{dx}\left(x\frac{d\Omega}{dx}\right) = 0,$$

$$(10)\qquad \frac{d}{dz}\left(\frac{1}{x}\frac{dM}{dz}\right) + \frac{d}{dx}\left(\frac{1}{x}\frac{dM}{dx}\right) = 0.$$

These are MAXWELL's results of § 703, obtained from independent physical considerations when x and z are replaced by his a and b; the symmetry round the axis z shows that the integration employed in MAXWELL's treatment can be effected, so that his result, where V and Ω are equivalent, may be written

$$(11)\qquad \frac{dM}{da} = \int_0^{2\pi} a \frac{dV}{dy} d\theta = 2\pi a \frac{dV}{dy},$$

$$(12)\qquad \frac{dM}{db} = -\int a \frac{dV}{dr} d\theta = -2\pi a \frac{dV}{dr}.$$

The potential of the circular disc is now written, either from (1) or (6), § 42,

$$V = 4a\int_v^\infty \left(1 - \frac{\mathrm{ch}^2 v \cos^2 w}{\mathrm{ch}^2 u} - \frac{\mathrm{sh}^2 v \sin^2 w}{\mathrm{sh}^2 u}\right) \frac{\mathrm{sh}\, u du}{\sqrt{[(\mathrm{ch}^2 u - \mathrm{ch}^2 v)\cdot(\mathrm{ch}\, u - \cos^2 w)]}}$$

$$(13) \quad \begin{aligned} &= 4\int_{s_1}^\infty \left[a - A\frac{\sqrt{(s_1 - s_3 \cdot s_2 - s_3)}}{s - s_3} - z\frac{\sqrt{(s_1 - \sigma \cdot \sigma - s_2)}}{s - \sigma}\right]\frac{\sqrt{(\sigma - s_3)}ds}{\sqrt{S}} \\ &= 4a\sqrt{\left(\frac{\sigma - s_3}{s_1 - s_3}\right)}K - 4A\sqrt{\left(\frac{\sigma - s_3}{s_2 - s_3}\right)}(K - H) - 4zA(fK') \\ &= \frac{4aK}{\mathrm{ch}\, v} - A\Omega' - z\Omega, \end{aligned}$$

of which each term can be interpreted by means of the homogeneity relation

$$(14) \qquad V = a\frac{dV}{da} + A\frac{dV}{dA} + z\frac{dV}{dz},$$

since

$$(15) \qquad \frac{dV}{dz} = -\Omega, \qquad \frac{dV}{dA} = -\Omega',$$

and therefore

$$(16) \qquad \frac{dV}{da} = \frac{4K}{\mathrm{ch}\, v} = \frac{8aK}{r_1 + r_2},$$

the potential at P of the circular ring round the rim of the disc.

Similarly in the case of the elliptic disc in § 40 the term in (8)

$$(17) \qquad 4ab\int \frac{d\epsilon}{\sqrt{E}} = \frac{4abK}{\sqrt{(\lambda - \nu)}}$$

may be interpreted as the potential at P of a ring round the ellipse cut off by a consecutive confocal ellipse, so that the thickness is inversely proportional to the perpendicular from the center on the tangent.

The Stokes function of a circular plate and ring.

44. Denote the Stokes function of the plate by N, using b again instead of z; then, as in §§ 31, 43,

$$(1) \qquad \frac{dN}{db} = 2\pi A\frac{dV}{dA} = -2\pi A\Omega' = -4\pi(r_1 + r_2)(K - H),$$

$$\begin{aligned} \frac{dN}{dA} &= -2\pi A\frac{dV}{db} = 2\pi A\Omega \\ (2) \qquad &= 2\pi A[2\pi(1 - f) - 4K\,\mathrm{zn}\, fK'] \\ &= 2\pi A \cdot 4A(fK'), \end{aligned}$$

and by analogy

$$(3)\qquad \frac{dN}{da} = 2\pi a\cdot 4A(1-f)K' = 2\pi a\left[2\pi f - 4K\mathrm{zn}(1-f)K'\right],$$

a change of f into $1-f$, and A into a; and this term is the Stokes function of the circular ring, of which the potential function is given in (1), § 33: so that the Stokes function at P of the circle AQB is $2\pi a$ times the apparent area at P'.

Then by the homogeneity of the second degree in a, A, b, of N

$$(4)\qquad \begin{aligned} 2N &= a\frac{dN}{da} + A\frac{dN}{dA} + b\frac{dN}{db} \\ &= 8\pi a^2 A(1-f)K' + 8\pi A^2\cdot AfK' - 4\pi b(r_1+r_2)(K-H), \end{aligned}$$

$$(5)\qquad N = 4\pi(a^2-A^2)B(fK') + 2\pi^2A^2 - 2\pi b(r_1+r_2)(K-H+K\mathrm{dn}^2 fK').$$

For points on the plate, $f=0$, $N=2\pi^2A^2$: while $f=1$ for points in its plane outside, and $N=2\pi^2a^2$; and $N-2\pi^2A^2$ may be taken as the Stokes function at P of this plate and a coaxial equal and opposite plate at the level of P.

Changing the superficial density from unity to $1/2\pi p$, we shall see in the sequel that

$$(6)\qquad \frac{2\pi^2A^2 - N}{2\pi p} = M,$$

where M is the function in § 54 investigated by Viriamu JONES for a helical current, or the equivalent cylindrical current sheet; and the potential of the two end plates is the magnetic potential of the cylinder magnetized longitudinally, while the Stokes function $N-2\pi^2A^2$ is the magnetic potential for magnetization across the axis of the cylinder.

The various dissections of an integral.

45. To arrive geometrically at the expression of the conical angle Ω by means of the angles θ, ϕ, ψ, χ, ω defined previously on fig. A and B, we employ the idea given by MAXWELL in E. and M. § 418, that the area Ω, cut out on unit sphere by a cone whose vertex is at the centre, is equal to 2π minus the length Φ of the curve traced out on the sphere by the reciprocal cone; this is a direct result and generalisation of the theorem that the sum of the area of a spherical triangle and the perimeter of the polar triangle is 2π.

The same idea is employed by SCHWARZ in his memoir *Die Scheinbare Grösse des Ellipsoides*, the apparent area and conical angle being the same thing, either of the ellipsoid, or of the tangent cone, or any elliptic section of it.

If the tangent plane PQY of the cone, shown in elevation and plan in figures A and B, turns through an angle $d\Phi$ while OQ turns through the small angle

$d\theta$, and if the normal line Pq of the cone cuts the plane AQB in q on YM produced (fig. B), then Q and q describe curves which are polar reciprocal with respect to M; so that QMy is perpendicular to qy the tangent at q; and the sector velocity of q round M

$$(1)\qquad \tfrac{1}{2}Mq^2\frac{d\theta}{dt}=\tfrac{1}{2}Pq^2\frac{d\Phi}{dt}\cos MyP,$$

$$(2)\qquad \frac{d\Phi}{d\theta}=\frac{Mq^2}{Pq^2}\frac{Py}{My}=\frac{PM^2}{PY^2}\cdot\frac{PQ}{PM}=\frac{1}{1-\dfrac{QY^2}{PQ^2}}\frac{PM}{PQ},$$

so that, in MAXWELL's notation, with $OM=A$,

$$(3)\qquad 2\pi-\Omega=\Phi=\int_0^{2\pi}\frac{1}{1-\dfrac{A^2\sin^2\theta}{PQ^2}}\cdot\frac{bd\theta}{PQ}.$$

Now from § 27

$$(4)\qquad \frac{A^2\sin^2\theta}{PQ^2}=\frac{4a^2A^2\sin^2\theta}{(r_1+r_2)^2CQ^2}=\left(\frac{2A}{r_1+r_2}\right)^2\sin^2\chi,\qquad \frac{bd\theta}{PQ}=\frac{2b}{r_1+r_2}\frac{d\chi}{\Delta\chi},$$

so that, in LEGENDRE's form of § 14,

$$(5)\qquad \Phi=\int_0^{\frac{1}{2}\pi}\frac{4\surd a}{1+n\sin^2\chi}\frac{d\chi}{\Delta\chi},$$

$$(6)\quad n=-\left(\frac{2A}{r_1+r_2}\right)^2,\quad k=\frac{r_1-r_2}{r_1+r_2},\quad a=(1+n)\left(1+\frac{k^2}{n}\right)=\left(\frac{2b}{r_1+r_2}\right)^2.$$

To reduce Φ to our form B in § 5, we put

$$(7)\qquad \sin^2\chi=\frac{s-s_3}{s_2-s_3},\qquad \frac{d\chi}{\Delta\chi}=\frac{\surd(s_1-s_3)\,ds}{\surd S},$$

and interpreted in fig. A, where $OM=A$, $OC=ak$, $OC'=a/k$,

$$(8)\quad \frac{\sigma-s_3}{s_2-s_3}=-\frac{1}{n}=\left(\frac{r_1+r_2}{2A}\right)^2,\quad \frac{\sigma-s_3}{s_1-s_3}=\text{dn}^2 fK'=\left(\frac{r_1-r_2}{2A}\right)^2=k\frac{r_1^2-r_2^2}{4A^2}=k\frac{a}{A}=\frac{OC}{OM},$$

$$(9)\qquad \frac{\sigma-s_2}{s_2-s_3}=\frac{(r_1+r)^2-4A^2}{4A^2},\qquad \frac{s_1-\sigma}{s_1-s_3}=\frac{4A^2-(r_1-r_2)^2}{4A^2},$$

$$(10)\qquad \frac{s_1-\sigma\cdot\sigma-s_2}{s_1-s_3\cdot s_2-s_3}=\frac{b^2}{A^2},$$

so that, as before in § 34,

$$(11)\qquad \Phi=4\int_{s_3}^{s_2}\frac{\frac{1}{2}\surd(-\Sigma)}{\sigma-s}\frac{ds}{\surd S}=4B(fK'),$$

$$(12)\qquad \Omega=2\pi-4B(fK')=4A(fK')=4\int_{s_1}^{\infty}\frac{\frac{1}{2}\surd(-\Sigma)}{s-\sigma}\frac{ds}{\surd S}.$$

46. If $d\alpha$ denotes an element of area at any point Q inside the circle, due to any mode of dissection

$$(1) \qquad V=\int\frac{d\alpha}{PQ}, \qquad \Omega=\int\frac{bd\alpha}{PQ^3},$$

and the integrals which arise in the determination of these potential functions change in appearance with the method of dissection employed by the use of each system of variables; so it will save repetition in the sequel if we cite here some preliminary lemmas.

I. The potential at a point F on its axis of a flat ring-shaped disc, bounded internally and externally by concentric circles on diameter ab and AB, is

$$(2) \qquad \frac{\text{area}}{\tfrac{1}{2}(FA+Fa)}=2\pi(FA-Fa),$$

and thence also the same for any fraction of the ring cut out by two radii at an angle θ, replacing 2π.

II. The potential at P of the line AB in fig. A is

$$(3) \quad \int_{-a}^{a}\frac{dx}{PQ}=\log\frac{PA+PB+AB}{PA+PB-AB}=2\,\text{th}^{-1}\frac{AB}{PA+PB}=2\,\text{th}^{-1}\frac{PA-PB}{2OM},$$

and when P is at F, the potential of AB is twice the potential of OA, and

$$(4) \qquad 2\,\text{th}^{-1}\frac{OA}{FA}.$$

In the elliptic coördinates of § 42 the potential at P assumes the various forms

$$(5) \quad \begin{aligned}\log\frac{\text{ch}\,v+1}{\text{ch}\,v-1}&=2\log\frac{\text{ch}\,v+1}{\text{sh}\,v}=2\log\frac{\text{sh}\,v}{\text{ch}\,v-1}=\int_v^\infty\frac{dv}{\text{sh}\,v}\\&=2\,\text{sh}^{-1}\frac{1}{\text{sh}\,v}=2\,\text{ch}^{-1}\frac{\text{ch}\,v}{\text{sh}\,v}=2\,\text{th}^{-1}\frac{1}{\text{ch}\,v}=\text{ch}^{-1}\frac{\text{ch}^2 v+1}{\text{ch}^2 v-1}, \text{ etc.}\end{aligned}$$

The Stokes function at P of the line AB is

$$(6) \qquad 2\pi(PA-PB)=4\pi a\cos w,$$

an expression much simpler.

III. The conical angle of the ring as seen from F on its axis is

$$(7) \qquad 2\pi\left(\frac{FO}{Fa}-\frac{FO}{FA}\right)=\frac{FO\times\text{area}}{FA\cdot Fa\cdot\tfrac{1}{2}(FA+Fa)},$$

and when the diameters ab, AB are nearly equal, this may be replaced by

$$(8) \qquad \frac{FO\times\text{area}}{FA^3}.$$

So also for any strip cut out by two radii; and if the radii make a small angle $d\theta$, the same expression holds when the plane of the strip is tilted about its mean radius OA through any angle, provided FO is replaced by the perpendicular FM on the plane, and the expression becomes

$$(9)\qquad \left(\frac{FM}{Fa}-\frac{FM}{FA}\right)d\theta.$$

IV. For a narrow parallel strip such as QQ'', of breadth dy, and PR the perpendicular on it from P, the conical angle at P is

$$(10)\qquad \frac{PM}{PR^2}(\cos PQR-\cos PQ''R)dy=\frac{PM\times \text{area}}{PR^2\cdot QQ''}\left(\frac{RQ}{PQ}-\frac{RQ''}{PQ''}\right).$$

47. Considering that $V+b\Omega$ is composed of complete I. and II. E. I.'s, the complete III. E. I. which arises in the determination of Ω will serve at the same time for V; and according to the method of dissection we shall find it depend on one of the five following forms, which we shall represent in the following notation, in which the III. E. I. is not restricted to be complete;

$$(1)\qquad \Omega(MQ)=\int_0^\theta\frac{aA\cos\theta+a^2}{MQ^2}\frac{bd\theta}{PQ}=\int\frac{a\cdot MY}{MQ^2}\frac{bd\theta}{PQ},$$

$$(2)\qquad \Omega(PY)=\int\frac{PQ^2}{PY^2}\frac{bd\theta}{PQ},$$

$$(3)\qquad \Omega(PZ)=\int\frac{aA\cos\theta+A^2+b^2}{PZ^2}\frac{bd\theta}{PQ},$$

where PZ is the perpendicular on OQ, and $PZ^2=A^2\sin^2\theta+b^2$;

$$(4)\qquad \Omega(PN)=\int\frac{QN^2}{PN^2}\frac{bd\theta}{PQ},$$

$$(5)\qquad \Omega(PR)=\int\frac{a\cos\theta(a\cos\theta+A)}{PR^2}\frac{bd\theta}{PQ}=\int\frac{ON\cdot MN}{PR^2}\frac{bd\theta}{PQ},$$

and these III. E. I.'s are connected in pairs by the ten equations following, as is verified by differentiation:

$$(6)\qquad \begin{aligned}\Omega(PY)-\Omega(PZ)=I_1&=\sin^{-1}\frac{MZ\cdot QZ}{PY\cdot PZ}=\cos^{-1}\frac{PM\cdot PQ}{PY\cdot PZ}\\ &=\tfrac{1}{2}\pi-\text{angle between planes } PQY,\ PQZ;\end{aligned}$$

$$(7)\qquad \begin{aligned}\Omega(MQ)-\Omega(PZ)=I_2&=\sin^{-1}\frac{MZ\cdot PQ}{PZ\cdot MQ}=\cos^{-1}\frac{MN\cdot MP}{PZ\cdot MQ}\\ &=\text{angle between } MPQ,\ QPZ;\end{aligned}$$

$$\begin{aligned}\Omega(MQ)-\Omega(PY)=I_3&=\sin^{-1}\frac{MZ\cdot MP}{PY\cdot MQ}=\cos^{-1}\frac{QZ\cdot PQ}{PY\cdot MQ}.\\&=\text{angle between } PQT,\ PQY;\end{aligned}\tag{9}$$

$$\begin{aligned}\Omega(MQ)+\Omega(PN)=I_4&=\sin^{-1}\frac{QN\cdot PM}{PN\cdot MQ}=\cos^{-1}\frac{MN\cdot PQ}{PN\cdot MQ}\\&=\text{angle between } PQT,\ QMR;\end{aligned}\tag{9}$$

$$\begin{aligned}\Omega(MQ)+\Omega(PR)=I_5&=\sin^{-1}\frac{QN\cdot PQ}{PR\cdot MQ}=\cos^{-1}\frac{MN\cdot PM}{PR\cdot MQ}\\&=\text{angle between } MPQ,\ MQR;\end{aligned}\tag{10}$$

$$\begin{aligned}\Omega(PR)-\Omega(PN)=I_6&=\sin^{-1}\frac{QN\cdot MN}{PR\cdot PN}=\cos^{-1}\frac{PM\cdot PQ}{PR\cdot PN}\\&=\tfrac{1}{2}\pi-\text{angle between } QPR,\ QPN;\end{aligned}\tag{11}$$

$$\begin{aligned}\Omega(PR)+\Omega(PY)=I_7&=\sin^{-1}=\frac{\sin\theta(aA\cos\theta+a^2+b^2)}{PR\cdot PY}\\&=\cos^{-1}\frac{PM\cos\theta\cdot PQ}{PR\cdot PY}\\&=\tfrac{1}{2}\pi-\text{angle between } QPR,\ QPY;\end{aligned}\tag{12}$$

$$\begin{aligned}\Omega(PR)+\Omega(PZ)=I_8&=\sin^{-1}\frac{PM\sin\theta\cdot PQ}{PR\cdot PZ}=\cos^{-1}\frac{aA\sin^2\theta-b^2\cos\theta}{PR\cdot PZ}\\&=\text{angle between } QPZ,\ QPR;\end{aligned}\tag{13}$$

$$\begin{aligned}\Omega(PY)-\Omega(PN)=I_9&=\sin^{-1}\frac{PM\sin\theta\cdot PQ}{PY\cdot PN}\\&=\cos^{-1}\frac{\sin\theta(aA\cos\theta+A^2+b^2)}{PY\cdot PN}\\&=\text{angle between } QPN,\ QPY;\end{aligned}\tag{14}$$

$$\begin{aligned}\Omega(PZ)+\Omega(PN)=I_{10}&=\sin^{-1}\frac{\sin\theta(aA\cos\theta+A^2+b^2)}{PZ\cdot PN}\\&=\cos^{-1}\frac{PM\cos\theta\cdot PQ}{PZ\cdot PN}\\&=\tfrac{1}{2}\pi-\text{angle between } QPN,\ QPZ.\end{aligned}\tag{15}$$

In a complete III. E. I., when θ ranges from 0 to 2π, the I term disappears or is 2π, and

$$(16)\qquad \begin{aligned}\Phi &= 2\pi - \Omega = \Omega(MQ) = \Omega(PY) = \Omega(PZ)\\ &= 2\pi - \Omega(PN) = 2\pi - \Omega(PR).\end{aligned}$$

We shall require in conjunction with $\Omega(PZ)$ the associated integral

$$(17)\qquad \Omega_1(PZ) = \int \frac{\surd(A^2+b^2)(a+A\cos\theta)}{PZ^2}\frac{bd\theta}{PQ},$$

and then

$$(18)\qquad \Omega_1(PZ)+\Omega(PZ) = \int \frac{a+\surd(A^2+b^2)}{\surd(A^2+b^2)-A\cos\theta}\frac{bd\theta}{PQ},$$

$$(19)\qquad \Omega_1(PZ)-\Omega(PZ) = \int \frac{a-\surd(A^2+b^2)}{\surd(A^2+b^2)+A\cos\theta}\frac{bd\theta}{PQ},$$

two III. E. I.'s, with parameter f_1F', f_2F', such that

$$(20)\qquad \operatorname{sn} f_1F' = \frac{a-\surd(A^2+b^2)}{r_2}, \qquad \operatorname{sn} f_2F' = \frac{r_1}{a+\surd(A^2+b^2)},$$

leading to a simple geometrical construction; and it is found that $2\pi - \Omega_1(PZ)$ is the conical angle subtended by the circle at P_1 on OP produced.

So too in conjunction with $\Omega(PR)$ we require

$$(21)\qquad \Omega_1(PQ) = \int \frac{\surd(a^2+b^2)(A+a\cos\theta)}{PR^2}\frac{bd\theta}{PQ},$$

$$(22)\qquad \Omega_1(PR)+\Omega(PR) = \int \frac{A+a\cos\theta}{\surd(a^2+b^2)-a\cos\theta}\frac{bd\theta}{PQ},$$

$$(23)\qquad \Omega_1(PR)-\Omega(PR) = \int \frac{A+a\cos\theta}{\surd(a^2+b^2)+a\cos\theta}\frac{bd\theta}{PQ},$$

and $\Omega_1(PR)$ is the conical angle at P_2 where P_1P_2 passes through B, or P_2P' is parallel to AB.

48. Thus with sector elements $\frac{1}{2}a^2d\theta$ for the determination of V, the potential dV at P of the element about OQ is given by

$$(1)\qquad \begin{aligned}\frac{dV}{d\theta} &= \int_0^a \frac{rdr}{PQ} = \int \frac{r+A\cos\theta}{PQ}d\theta - \int \frac{A\cos\theta d\theta}{PQ}\\ &= PQ - PO - 2A\cos\theta \operatorname{th}^{-1}\frac{a}{PQ+PO},\end{aligned}$$

of which the first part is by Lemma I the potential of the strip $OQq'o$, between OQ, oq' radiating from Z, and the second is by Lemma II the potential of a shaving of uniform breadth $d\theta \cdot A\cos\theta$ and length OQ.

As required in integration by parts, the second part of (1) can be written

$$2\frac{d}{d\theta}\left(A\sin\theta\,\mathrm{th}^{-1}\frac{a}{PQ+PO}\right)-\frac{a^2A^2\sin^2\theta}{PQ\cdot PO+aA\cos\theta+A^2+b^2}\,\frac{1}{PQ}$$

$$(3)\qquad \begin{aligned} &=2\frac{d}{d\theta}\left(A\sin\theta\,\mathrm{th}^{-1}\frac{a}{PQ+PO}\right)-\frac{PQ\cdot PO-aA\cos\theta-A^2-b^2}{PZ^2}\,\frac{A^2\sin^2\theta}{PQ}\\ &=2\frac{d}{d\theta}\left(A\sin\theta\,\mathrm{th}^{-1}\frac{a}{PQ+PO}\right)-PO\left(1-\frac{b^2}{PZ^2}\right)\\ &\qquad\qquad+\frac{aA\cos\theta+A^2+b^2}{PQ}-\frac{bd\Omega(PZ)}{d\theta}\,; \end{aligned}$$

So also, if $d\Omega$ is the conical angle subtended by the sector element about OQ, the application of Lemmas III and IV shows that

$$(3)\qquad \begin{aligned} \frac{d\Omega}{d\theta}&=\frac{PM}{PO}-\frac{PM}{PQ}-\frac{PM}{PZ^2}\left(\frac{QZ}{PQ}-\frac{OZ}{PO}\right)A\cos\theta\\ &=b\frac{PO}{PZ^2}-\frac{d\Omega(PZ)}{d\theta}=b\frac{PO}{PZ^2}+\frac{dI_1}{d\theta}-\frac{d\Omega(PY)}{d\theta}, \end{aligned}$$

so that

$$(4)\qquad \frac{dV}{d\theta}+b\frac{d\Omega}{d\theta}=\frac{a^2+aA\cos\theta}{PQ}-2\frac{d}{d\theta}\left(A\sin\theta\,\mathrm{th}^{-1}\frac{a}{PQ+PO}\right)$$

and thence the previous expressions in (13), § 43.

49. The dissection which leads directly to MINCHIN's integral in the Philosophical Magazine, February, 1894, is made by lines radiating from M.

Denoting the angle AMQ by η,

(1) $\frac{1}{2}MQ^2d\eta =$ sector element about $MQ = \frac{1}{2}MY\cdot ad\theta = \frac{1}{2}(a^2+aA\cos\theta)d\theta$,

and by Lemma III,

$$(2)\qquad d\Omega=\left(1-\frac{PM}{PQ}\right)d\eta,$$

$$(3)\qquad \frac{d\Omega}{d\theta}=\frac{d\eta}{d\theta}-\frac{a^2+aA\cos\theta}{MQ^2}\,\frac{b}{PQ}=\frac{d\eta}{d\theta}-\frac{d\Omega(MQ)}{d\theta},$$

so that for any incomplete arc AQ

$$(4)\qquad \Omega=\eta-\Omega(MQ).$$

Now with the substitution of § 26 and in addition

$$(5)\qquad MQ^2=m^2(\tau-t),$$

so that

$$(6)\qquad b^2=m^2(t_1-\tau),\quad (a\sim A)^2=m^2(\tau-t_2),\quad (a+A)^2=m^2(\tau-t_3),$$

then when M is inside the circle

$$(7)\qquad a - A = m\surd(\tau - t_2), \qquad a = \tfrac{1}{2}m\left[(\tau - t_2) + \surd(\tau - t_3)\right],$$

$$(8)\qquad aA\cos\theta = \tfrac{1}{4}m^2(t_2 + t_3 - 2t),$$

$$(9)\qquad aA\cos\theta + a^2 = \tfrac{1}{2}m^2\left[\tau - t + \surd(\tau - t_2\cdot\tau - t_3)\right].$$

Integrating round the circle, the angle η makes a complete revolution when M is inside, and

$$\Omega = 2\pi - \int_{t_3}^{t_2}\left[1 + \frac{\surd(\tau - t_2\cdot\tau - t_3)}{\tau - t}\right]\frac{\surd(t_1 - \tau)\,dt}{\frac{1}{2}\surd T}$$

$$(10)\qquad = 2\pi - 2\sqrt{\frac{t_1 - \tau}{t_1 - t_3}}\int\frac{\surd(t_1 - t_3)\,dt}{\surd T} - 2\int\frac{\frac{1}{2}\surd(-T')}{\tau - t}\frac{dt}{\surd T}$$

$$= 2\pi - 2Fc'\,\mathrm{sn}\,2fF' - 2B(2fF')$$

$$= 2\pi(1 - f) - 2F\,\mathrm{zn}\,2fF' - 2Fc'\,\mathrm{sn}\,2fF';$$

hence MINCHIN's result obtained directly.

If M is outside the circle, the angle η oscillates; but we must take

$$(11)\qquad a = \tfrac{1}{2}m\left[\surd(\tau - t_3) - \surd(\tau - t_2)\right],$$

and now

$$(12)\quad \Omega = 2\int\left[\frac{\surd(\tau - t_2\cdot\tau - t_3)}{\tau - t} - 1\right]\frac{dt}{\surd T} = 2\pi f + 2F\,\mathrm{zn}\,2fF' - 2Fc'\,\mathrm{sn}\,fF',$$

equivalent to a change of f into $1 - f$, as in going from P to P'' so that (10) may be taken to hold for all cases.

As P travels round the circle CPC' the quantities c, k, F, E, K, H do not vary, and f may be taken as the independent variable.

Starting from C in the plane of the disc, where $f = 0$,

$$(13)\qquad \Omega = 2\pi, \qquad V = 4a\frac{E + c'F}{1 + c'} = 4aH, \qquad \frac{V}{AC} = 2(E + c'F).$$

At C' outside the disc in its plane, $f = 1$,

$$(14)\qquad \Omega = 0, \qquad V = 4a\frac{E - c'F}{1 - c'} = 4a\frac{H - k'^2K}{k}, \qquad \frac{V}{AC'} = 2(E - c'F).$$

As P travels to P'', f increases from f to $f'' = 1 - f$, so that

$$(15)\qquad \Omega'' = 2\pi f + 2F\,\mathrm{zn}\,2fF' - 2Fc'\,\mathrm{sn}\,2fF',$$

$$(16)\quad \Omega + \Omega'' = 2\pi - 4Fc'\,\mathrm{sn}\,2fF' = 2\pi - 4F\sin\psi', \qquad \Omega - \Omega'' = 2\pi - 4B(2fF'),$$

and Ω'' is the apparent area of the coaxial circle through P seen from any point on the circle AB.

$$(17)\qquad \frac{V}{AP}+\frac{V''}{AP''}=4E-4F\sin^2\psi'-(\Omega+\Omega'')\sin\psi'=4E-2\pi\sin\psi',$$

which is MINCHIN's equation (23), p. 212, Philosophical Magazine, February, 1894.

When P reaches P', f has increased to $f'=1+f$,

$$(18)\qquad \Omega'=-2\pi f+2F\,\mathrm{zn}\,2fF'+2F\sin\psi',$$

$$(19)\qquad \Omega-\Omega'=2\pi,$$

but

$$(20)\qquad V'-V''=0.$$

At P''', f has reached $f'''=2-f$;

$$(21)\qquad \Omega-\Omega'''=4\pi,\qquad V-V'''=0,$$

so that V has no cyclic constant as P describes a curve linked with the circular edge; a cylic constant like the 4π for Ω would imply a non-conservative potential and the possibility of the creation of energy, called otherwise Perpetual Motion.

50. Dissected by QN into strips $ydx=a\sin^2\theta d\theta$, then by Lemma IV, § 46,

$$(1)\qquad d\Omega=\frac{bdx}{PN^2}\,\frac{a\sin\theta}{PQ}=\frac{a^2\sin^2\theta}{PN^2}\,\frac{bd\theta}{PQ},\qquad \frac{d\Omega}{d\theta}=\frac{d\Omega(PN)}{d\theta},$$

so that χ of § 27 is the appropriate coördinate angle, leading to E. F.'s of hK and fK', and

$$(2)\qquad \begin{aligned}\Omega&=\int_0^{2\pi}\frac{a^2\sin^2\theta}{\left(\frac{r_1+r_2}{2a}\right)^2CQ^2-a^2\sin^2\theta}\,\frac{bd\theta}{PQ}\\&=\int^{\frac12\pi}\frac{4\sin^2\chi}{\left(\frac{r_1+r_2}{2a}\right)^2-\sin^2\chi}\,\frac{b}{r_1+r_2}\,\frac{d\chi}{\Delta\chi},\end{aligned}$$

which is reduced to the form A of § 5 by putting

$$(3)\qquad \sin^2\chi=\frac{s_1-s_3}{s-s_3},\qquad \left(\frac{r_1-r_2}{2a}\right)^2=\frac{s_1-s_3}{\sigma-s_3},$$

and this makes

$$(4)\quad \Omega=4\int_{s_1}^{\infty}\frac{\frac12\sqrt{(-\Sigma)}}{s-\sigma}\,\frac{ds}{\sqrt{S}}=4A(fK')=2\pi(1-f)-4K\,\mathrm{zn}\,fK'.$$

With the variable y of § 29

$$(5) \qquad 1-y=\frac{PN^2}{PQ^2}, \qquad \frac{y}{1-y}=\frac{a^2\sin^2\theta}{PN^2},$$

and denoting the value of Y for $y=1$ by Y',

$$(6) \qquad Y'=4(y_1-1)(y_2-1)=-4\frac{b^2}{A^2}=-4\frac{b^2}{a^2}y_1y_2,$$

so that from (11) § 29

$$(7) \qquad \begin{aligned} \Omega &= \int_0^{y_2}\frac{2y}{1-y}\,\frac{b}{a}\,\frac{\sqrt{(y_1y_2)}\,dy}{\sqrt{Y}}=\int\left(\frac{1}{1-y}-1\right)\frac{2\sqrt{(-Y')}\,dy}{\sqrt{Y}} \\ &= 4B(f'K')-2K\sqrt{\frac{-Y'}{y_1}}, \end{aligned}$$

in which

$$(8) \qquad \operatorname{dn}^2 f'K'=\frac{\sigma'-s_3}{s_1-s_3}=\frac{1}{y_1}=\left(\frac{2A}{r_1+r_2}\right)^2, \qquad \operatorname{dn}^2 fK'=\left(\frac{r_1-r_2}{2A}\right)^2,$$

from (8) § 45,

$$(9) \qquad \operatorname{dn}^2 f'K'\operatorname{dn}^2 fK'=k^2, \qquad f+f'=1,$$

$$(10) \qquad \begin{aligned} \Omega &= 2\pi(1-f)+4K\operatorname{zn}(1-f)K'-4K\frac{2b}{r_1+r_2} \\ &= 2\pi(1-f)-4K\operatorname{zn} fK', \end{aligned}$$

as before in (4).

So also by Lemma II

$$(11) \qquad dV=dx\tfrac{1}{2}\log\frac{PQ+a\sin\theta}{PQ-a\sin\theta}=a\sin\theta d\theta\operatorname{th}^{-1}\frac{a\sin\theta}{PQ},$$

and as before in § 48 we may transform this into

$$(12) \quad \frac{dV}{d\theta}=-\frac{d}{d\theta}\left[(a\cos\theta+A)\operatorname{th}^{-1}\frac{a\sin\theta}{PQ}\right]+\frac{a^2+aA\cos\theta}{PQ}-b\frac{d\Omega(PN)}{d\theta},$$

so that, as before in (4) § 48, over the circle,

$$(13) \qquad V+b\Omega=\int_0^{2\pi}(a+A\cos\theta)\frac{ad\theta}{PQ}=a\frac{dV}{da}+A\frac{dV}{dA}.$$

To connect with the E. F.'s of hF and $2fF'$, by means of $\Omega(MQ)$,

$$(14) \qquad \frac{d\Omega}{d\theta}=\frac{d\Omega(PN)}{d\theta}=\frac{dI_4}{d\theta}-\frac{d\Omega(MQ)}{d\theta},$$

$$(15) \qquad \Omega=2\pi-\Omega(MQ),$$

as in (4) § 49.

51. Dissecting by lines QQ'' parallel to AB into strips

$$(1)\qquad QQ''dy = 2a^2 \cos^2 \theta d\theta,$$

of which by Lemma IV

$$(2)\quad d\Omega=\frac{bdy}{PR^2}\left(\frac{QR}{PQ}\pm\frac{RQ''}{PQ''}\right)=\left(\frac{a^2\cos^2\theta+aA\cos\theta}{PQ}+\frac{a\cos^2\theta-aA\cos\theta}{PQ''}\right)\frac{bd\theta}{PR^2},$$

we may take

$$(3)\qquad \frac{d\Omega}{d\theta}=\frac{a^2\cos^2\theta+aA\cos\theta}{PR^2}\,\frac{b}{PQ}=\frac{d\Omega(PR)}{d\theta},$$

and then integrate round the circle from 0 to 2π.

So also from Lemma II, with the same dissection,

$$(4)\ dV=dy\log\frac{PQ+PQ''+2a\cos\theta}{PQ+PQ''-2a\cos\theta}=2dy\,\text{th}^{-1}\frac{2a\cos\theta}{PQ+PQ}=2dy\,\text{th}^{-1}\frac{PQ-PQ''}{2A},$$

$$\frac{dV}{d\theta}-2\frac{d}{d\theta}\left(y\,\text{th}^{-1}\frac{PQ-PQ''}{2A}\right)=\frac{4a^2A^2\sin^2\theta}{4A^2-(PQ-PQ'')^2}\left(\frac{1}{PQ}+\frac{1}{PQ''}\right)$$

$$(5)\qquad =\tfrac{1}{2}a^2\sin^2\theta\,\frac{PQ\cdot PQ''-A^2+a^2-b^2}{PR}\left(\frac{1}{PQ}+\frac{1}{PQ''}\right)$$

$$=(PR^2-b^2)\left(\frac{a^2+aA\cos\theta+b^2}{PR^2\cdot PQ}+\frac{a^2-aA\cos\theta+b^2}{PQ^2\cdot PR'}\right),$$

and here again, in integrating round the circle we can take

$$(6)\qquad V=\int_0^{2\pi}(PR^2-b^2)\,\frac{a^2+aA\cos\theta+b^2}{PR^2}\,\frac{d\theta}{PQ},$$

and $V+b\Omega$ is the same as before in (13) § 50.

52. Finally there is the integration for V and Ω required when the circle is dissected into concentric rings; and now dV/dr is the potential of a circle of radius r, and

$$(1)\qquad \frac{dV}{da}=4F\frac{a}{r_1}=8K\frac{a}{r_1+r_2},$$

$$(2)\qquad V=\int_0^a 4F\frac{a}{r_1}\,da=\int 8K\frac{a}{r_1+r_2}\,da,$$

of which the result is known already by the other dissections, although the integration appears intractable at first sight.

So also, from (3) § 31

$$(3)\qquad \frac{d\Omega}{da}=-\frac{d^2V}{da\,db}=\frac{4ab}{r_1^3}\,\frac{E}{c'^2}=\frac{32ab}{(r_1+r_2)^3}\,\frac{2H-k'^2K}{k'^4},$$

leading to an integral of appearance still more intractable.

53. Treating the potential U of the solid cylinder in the preceding manner, as given by a function homogeneous in the second degree of a, A, b,

$$(1)\qquad 2U = a\frac{dU}{da} + A\frac{dU}{dA} + b\frac{dU}{db},$$

the first term is a times the potential of the curved surface, or skin potential as it may be called of the cylinder, so that

$$(2)\qquad \frac{1}{a}\frac{dU}{da} = \int_0^{2\pi} \mathrm{th}^{-1}\frac{b}{PQ}\,d\theta = \int_{t_3}^{t_2} 2\,\mathrm{th}^{-1}\sqrt{\frac{t_1-\tau}{t_1-t}}\cdot\frac{dt}{\sqrt{(t_2-t\cdot t-t_3)}},$$

not an elliptic integral; but dU/dA and dU/db are given in §§ 44, 54 as the potential of the cylinder due to transverse and longitudinal magnetization.

We shall apply the same method in § 64 to Mr. Coleridge FARR's problem (Proceedings of the Royal Society, November, 1898), to determine the electro-magnetic force of a cylindrical coil of finite thickness, where a similar intractable integral is encountered, in conjunction with elliptic integrals, which we consider tractable.

A graphical representation can be given on the Mercator chart of the integral dW/ada in (2) by denoting the angle MQP by ϕ, so that

$$(3)\qquad \mathrm{th}^{-1}\frac{b}{PQ} = \mathrm{th}^{-1}\sin\phi = \log\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi\right);$$

then if ϕ, θ are taken as latitude and longitude, and the curve drawn on the Mercator chart, the area will represent the integral (2).

Or else $\mathrm{th}^{-1}(b/PQ)$ can be expanded in odd powers of $b/PQ = b/(r_1\Delta\omega)$, and then the integral of each term such as

$$(4)\qquad D_n = \int_0^{\frac{1}{2}\pi}\frac{d\omega}{(\Delta\omega)^{2\pi+1}}$$

can be expressed by means of F and E, by means of the recurring relation

$$(5)\qquad (2n-1)c'^2D_n - (2n-2)(1+c'^2)/D_{n-1} + (2n-3)D_{n-2} = 0,$$

starting with $D_0 = F$, $D_1 = E/c'^2$.

Mutual induction of a circle and coaxial helix.

54. The III. E. I. arises in the calculation of M between a helix and a coaxial ring; this has been carried out by Viriamu JONES in the Philosophical Magazine, January, 1889; Philosophical Transactions (1891); Proceedings of the Royal Society, December, 1897; consult also WEBSTER, *Electricity and Magnetism*, p. 457; MASCART and JOUBERT, *Electricity* (translated by ATKINSON); and GRAY, *Absolute Measurements in Electricity*, volume II, p. 313.

It results from the form of the integral that M is the same as between the ring and a uniform current sheet in a coaxial cylinder replacing the helix, in which the linear helical current γ_h and the superficial current γ across unit height of the cylinder are connected with p the pitch of the helix by the relation

$$(1) \qquad \gamma_h = 2\pi p \gamma.$$

For if N denotes the number of windings and Θ the total angle of the helix of height b,

$$(2) \qquad N\gamma_h = b\gamma, \qquad \text{where} \qquad \Theta = \frac{b}{p}, \qquad N = \frac{\Theta}{2\pi} = \frac{b}{2\pi p}.$$

The potential of this current sheet has been worked out by MINCHIN in the Philosophical Magazine, February, 1894, as being the same as the potential of a cylinder magnetized longitudinally, or of two end plates of the cylinder, positive and negative; the expression for the potential of a plate has been cited already in § 43 above.

We notice now that M is the Stokes function of the two end plates, or of the cylinder magnetized longitudinally, and that $M\gamma_h/2\pi A$ is the magnetic potential in the plane $x0z$ at a distance A from the axis of the cylinder magnetized transversely parallel to this plane with intensity γ; and denoting this magnetic potential by $\Omega'\gamma$, and by W the potential of the solid cylinder,

$$(3) \qquad \frac{M\gamma_h}{2\pi A} = \Omega'\gamma, \qquad \text{or} \qquad Mp = A\Omega';$$

and

$$(4) \quad A\Omega' = -A\frac{dU}{dx} = \int_0^b \int_0^{2\pi} \frac{-aA\cos\theta\, d\theta\, dz}{\sqrt{(\alpha^2+z^2)}} = \int_0^{2\pi} -aA\cos\theta\, d\theta\, \text{th}^{-1}\frac{b}{PQ},$$

the first integral expression given by Viriamu JONES, employing his notation of α^2 and α^2+z^2 for MQ^2 and PQ^2, but using z instead of x, and changing his θ into $\pi - \theta$, to agree with the preceding treatment.

The result for Mp is the same whether $2a$ denotes the diameter of the helix and $2A$ of the ring, or vice-versa as with Viriamu JONES, being symmetrical in a and A.

Integration by parts enables him to replace (4) by

$$(5) \quad A\Omega' = \int_0^{2\pi} \frac{a^2A^2b\sin^2\theta\, d\theta}{\alpha^2\sqrt{(\alpha^2+b^2)}} = \tfrac{1}{2}\int(-aA\cos\theta + a^2)\frac{bd\theta}{PQ} - \tfrac{1}{2}(a^2-A^2)\,\Omega(MQ),$$

integrals discussed already in §§ 33, 49; and (5) is the equivalent of the double integral

$$(6) \qquad A\Omega' = \int_0^b \int_0^{2\pi} \frac{a^2A^2\sin^2\theta d\theta dz}{(\alpha^2+z^2)^{\frac{3}{2}}} = \int_0^b \int_0^{\pi} \frac{2a^2\, d\theta\, dz}{PQ^3},$$

showing that the magnetic potential Ω' at P of the cylinder magnetized parallel to OM is the same as that of the surface magnetized circumferentially with intensity $a\sin\theta$.

It is the second form (5) which is reducible immediately to the standard elliptic form, and by the substitution in §§ 26, 49, writing

$$(7) \qquad MQ^2 = \alpha^2 = A^2 + 2Aa\cos\theta + a^2 = m^2(\tau - t),.$$

$$(8) \qquad PQ^2 = \alpha^2 + b^2 = m^2(t_1 - t),$$

$$(9) \quad b^2 = m^2(t_1 - \tau), \qquad (a \sim A)^2 = m^2(\tau - t_2), \qquad (a + A)^2 = m^2(\tau - t_3),$$

and thence from (5)

$$(10) \qquad A\Omega' = \tfrac{1}{2}b\int_{t_3}^{t_2} \frac{m^4(t_2 - t)(t - t_3)}{m^2(\tau - t)} \frac{\dfrac{dt}{\surd(t_2 - t\cdot t - t_3)}}{m\surd(t_1 - t)}$$

$$= mb\int\frac{(t_2 - t)(t - t_3)\,dt}{(\tau - t)\surd T},$$

$$pM = A\Omega' = mb\int\left(t - t_3 + \tau - t_2 - \frac{\tau - t_2\cdot\tau - t_3}{\tau - t}\right)\frac{dt}{\surd T}$$

$$(11) \qquad = r_1 b\left[\int\frac{t - t_3}{\surd(t_1 - t_3)}\frac{dt}{\surd T} + \frac{\tau - t_2}{t_1 - t_3}\int\frac{\surd(t_1 - t_3)\,dt}{\surd T}\right.$$

$$\left. - \sqrt{\frac{\tau - t_2\cdot\tau - t_3}{t_1 - \tau\cdot t_1 - t_3}}\int\frac{\frac{1}{2}\surd(-T')}{\tau - t}\frac{dt}{\surd T}\right].$$

Hence, on fig. A,

$$(12) \qquad \sqrt{\frac{t_1 - \tau}{t_1 - t_2}} = \frac{b}{r_2} = \sin\psi, \qquad \sqrt{\frac{\tau - t_2}{t_1 - t_2}} = \frac{a - A}{r_2} = \cos\psi,$$

$$\sqrt{\frac{\tau - t_3}{t_1 - t_3}} = \frac{a + A}{r_1} = \cos\psi' = \Delta\psi, \qquad \psi = \operatorname{am} 2fF',$$

so that

$$(13) \qquad pM = r_1 b\,[F - E + Fc'^2\cos^2\psi - \cot\psi\cos\psi' B(2fF')].$$

$$(14) \qquad \frac{M}{r_1\Theta} = F - E - \cot\psi\cos\psi' A(1 - 2f)F',$$

the equivalent of Viriamu JONES's result in (6) p. 198, *Proceedings of the Royal Society*, 1898.

We shall replace his angle β by ψ_1 to avoid confusion with MAXWELL's modular angle β, and then

$$(15) \quad \psi_1 = \operatorname{am}(1 - 2f)F', \quad \sin\psi_1' = \frac{\tan\psi'}{\tan\psi} = \frac{a - A}{a + A}\frac{r_1}{r_2}, \quad \cos\psi_1' = \frac{\surd(aA)}{\frac{1}{2}(a + A)},$$

so that the angle ψ_1' is constructed on fig. A by drawing the circle on the diameter AM cutting the axis OF in G, and then drawing OP_1 perpendicular to the radius GO' of this circle, then the angle OAP_1 or OGO' is ψ_1'.

55. With the notation of § 47, equation (5) § 54, is written

$$(1)\quad 2A\Omega' + (a^2 - A^2)\Omega(MQ) = \int (a^2 - aA\cos\theta)\frac{bd\theta}{PQ}.$$

$$(2)\quad 2A\Omega' + (a^2 - A^2)\Omega(PY) = \int (a^2 - aA\cos\theta)\frac{bd\theta}{PQ} - (a^2 - A^2)I_3;$$

and in a complete revolution of θ from 0 to 2π,

$$(3)\quad \begin{aligned}\Omega(MQ) &= \int_0^\pi \frac{MQ^2 + a^2 - A^2}{MQ^2}\frac{bd\theta}{PQ} = b\int\frac{d\theta}{PQ} + \int\frac{a^2 - A^2}{MQ^2}\frac{bd\theta}{PQ}\\ &= 2F\frac{b}{r_1} + 2B(2fF') = 2Fc'\operatorname{sn} 2fF' + 2B(2fF')\end{aligned}$$

equivalent to the result in (10) § 49.

Also I_3 vanishes in a complete revolution of θ, and from (16) § 47,

$$\Omega(PY) = \Phi = 4B(fK'),$$

so that

$$(4)\quad \begin{aligned}2A\Omega' + 4(a^2 - A^2)B(fK') &= \int_0^{2\pi} (a^2 - aA\cos\theta)\frac{bd\theta}{PQ}\\ &= \frac{8a^2bK}{r_1 + r_2} + 2b(r_1 + r_2)(K - H),\end{aligned}$$

in which

$$(5)\quad \frac{2a}{r_1 + r_2} = \frac{1}{\operatorname{ch} v} = \operatorname{dn} fK',$$

$$(6)\quad \begin{aligned}pM = A\Omega' = b(r_1 + r_2)K\operatorname{dn}^2 fK' + b(r_1 + r_2)(K - H)\\ - 2(a^2 - A^2)B(fK'),\end{aligned}$$

in the form given in (5) § 44, the quadric transformation of (13) § 54.

This again, by means of (2), (3), § 17, can be transformed into

$$(7)\quad \frac{M}{(r_1 + r_2)\Theta} = K - H + \frac{\operatorname{cn} 2fK'}{\operatorname{sn} 2fK'}C(1 - 2f)K'.$$

In the indeterminate case of $2f = 1$ and $C(1 - 2f)K' = \infty$, we have $A = a$; but returning to the integral in (10) § 54, and putting $\tau = t_2$,

$$(8)\quad pM = A\Omega' = \frac{mb}{a}\int\frac{(t - t_3)dt}{\sqrt{T}} = bm\sqrt{(t_1 - t_3)}(F - E) = r_1b(F - E),$$

$$(9)\quad \frac{M}{\Theta} = r_1(F - E) = (r_1 + r_2)(K - H + kK),$$

and this is the coefficient of self induction of the helix on a final turn of itself.

Numerical calculation of the mutual induction.

56. The numerical calculation of M for given dimensions a, A, b now requires the five operations following, in the numerical tables of F. E., vol. II, LEGENDRE, and it is useful to check the calculation on fig. A, drawn to scale.

I. Calculate ψ and ψ' from

$$(1)\qquad \tan\psi = \frac{b}{a-A}, \qquad \tan\psi' = \frac{b}{a+A};$$

and then

$$r_1 = (a+A)\sec\psi', \qquad r_2 = (a-A)\sec\psi.$$

II. Calculate ψ_1 and the modular angle γ from

$$(2)\qquad \sin\psi_1 = \frac{\cos\psi}{\cos\psi'}, \qquad \cos\gamma = \frac{\sin\psi'}{\sin\psi}.$$

III. Calculate f by LEGENDRE's Table IX from

$$(3)\qquad 2f = \frac{F(\psi, c')}{F(\frac{1}{2}\pi, c')}, \qquad 1-2f = \frac{F(\psi_1, c')}{F(\frac{1}{2}\pi, c')}.$$

IV. Calculate $\operatorname{zn}(1-2f)F'$ by Table IX from

$$(4)\qquad \operatorname{zn}(1-2f)F' = E(\beta', c') - (1-2f)E(\tfrac{1}{2}\pi, c').$$

V. Look out

$$(5)\qquad F = F(\tfrac{1}{2}\pi, c), \qquad E = E(\tfrac{1}{2}\pi, c).$$

When f is one of the rational fractions of (1) § 4, we can express c, c', $\operatorname{sn} 2fF'$, $\operatorname{zn} 2fF'$, $\cdots$, as algebraical functions of a parameter, leaving F and E as the only transcendents, and these are tabulated by LEGENDRE in his Table I to an extra degree of accuracy.

In the numerical application of Viriamu JONES, Philosophical Transactions (1891), Proceedings of the Royal Society (1897),

$$2a = 21.02673, \qquad 2A = 13.01997, \qquad 2b = 5.02480 \text{ (inches)},$$

$$a - A = 4.00338, \qquad a + A = 17.02335, \qquad b = 2.51240;$$

and working with four figure logarithms we find $\psi = 32^\circ\, 3'$, $\psi' = 8^\circ\, 24'$, $\gamma = 74^\circ\, 1'$, $\psi_1 = 58^\circ\, 58'$; or to the nearest degree, $\gamma = 74^\circ$, $\psi_1 = 59^\circ$, thus avoiding proportional parts in Legendre's Table IX; $F = 2.70807$, $E = 1.08443$, and $F' = 1.60198$, $E(\psi_1, c') = 1.04123$, $1 - 2f = 0.6500$, $f = 0.1750$, $\pi f = 0.54985$, $E' = 1.54052$, $(1-2f)E' = 1.00134$, $E\psi_1 = 1.01847$, $\operatorname{zn}(1-2f)F' = 0.01713$, $F\operatorname{zn}(1-2f)F' = 0.04638$, $A(1-2f)F' = 0.50347$, $\cot\psi\cos\psi' A(1-2f)F' = 0.7952$, $F - E = 1.6236$, $M/r_1\Theta = 0.8284$, $\Theta = 201\pi$, $r_1 = 17.21$, $M = 9001$ inches, as against 9028 inches, given by Viriamu JONES.

This rough calculation is retained for the purpose of showing the term requiring extra calculated figures, compared with the parts where fewer figures are retained for the same degree of accuracy; thus of the three chief terms of $M/r_1\Theta$, the first $F - E = 1.6236$ is about double the second $\pi f \cot \psi \cos \psi' = 0.8638$, while the remainder $\cot \psi \cos \psi' F \operatorname{zn}(1 - 2f) F' = 0.0734$.

The modular angle must thus be determined with extra accuracy for the determination of $F - E$; and in this case raising the modular angle by one minute to 74° 1′ will make $F - E = 1.6258$, and raise M to 9024.

A diminution of 0.1 per cent. in f in the second term will raise M about 0.05 per cent. to about 9028.5, without affecting the last term appreciably.

Variation of the induction due to change in the dimensions.

57. Viriamu JONES calculates the effect of small error of measurement dA, da, db in A, a, b in giving a change dM in M, such that

$$(1) \qquad dM = \left(A\frac{\partial M}{\partial A}\right)\frac{dA}{A} + \left(a\frac{\partial M}{\partial a}\right)\frac{da}{a} + \left(b\frac{\partial M}{\partial b}\right)\frac{db}{b},$$

so that 1 per cent. change in A, a, b gives a change in M of

$$(2) \qquad \frac{A}{100}\frac{\partial M}{\partial A}, \qquad \frac{a}{100}\frac{\partial M}{\partial a}, \qquad \frac{b}{100}\frac{\partial M}{\partial b};$$

and their sum is $M/100$, because M is a homogeneous function of the variables A, a, b, and of the first degree, the total helical angle Θ or number of windings $\Theta \div 2\pi$ being kept constant. But if the pitch p of the helix is kept constant, M is of the second degree in A, a, b.

Differentiating the double integral expression for M in (4) § 54,

$$(3) \qquad M = \frac{\Theta}{b}\int_0^b \int_0^{2\pi} - \frac{Aa\cos\theta\, d\theta\, dz}{\sqrt{(\alpha^2 + z^2)}},$$

and supposing the limits of θ to be 0 and 2π in the sequel, we have

$$(4) \quad \frac{\partial M}{\partial A} = \frac{\Theta}{b}\int\int \frac{-a\cos\theta\, d\theta\, dz}{\sqrt{(\alpha^2 + z)}} + \frac{\Theta}{b}\int\int \frac{(A + a\cos\theta)Aa\cos\theta\, d\theta\, dz}{(\alpha^2 + z^2)^{\frac{3}{2}}},$$

$$(5) \quad A\frac{\partial M}{\partial A} = M + \Theta\int \frac{(A^2 + Aa\cos\theta)Aa\cos\theta\, d\theta}{\alpha^2\sqrt{(\alpha^2 + b^2)}} = \Theta A^2 \int \frac{a^2 + aA\cos\theta}{MQ^2}\frac{d\theta}{PQ},$$

and similarly

$$(6) \quad a\frac{\partial M}{\partial a} = M + \Theta\int \frac{(Aa\cos\theta + a^2)Aa\cos\theta\, d\theta}{\alpha^2\sqrt{(\alpha^2 + b^2)}} = \Theta a^2 \int \frac{aA\cos\theta + A^2}{MQ^2}\frac{d\theta}{PQ},$$

so that

$$(7)\qquad a\frac{dM}{da}+A\frac{dM}{dA}=2M-\Theta\int\frac{-aA\cos\theta d\theta}{PQ},$$

while, keeping Θ constant,

$$(8)\qquad bM=\Theta\int_0^b\int\frac{-aA\cos\theta d\theta dz}{\sqrt{(a^2+z^2)}},$$

$$(9)\qquad b\frac{dM}{db}+M=\Theta\int\frac{-aA\cos\theta d\theta}{PQ},$$

and then

$$(10)\qquad a\frac{dM}{da}+A\frac{dM}{dA}+b\frac{dM}{db}=M,$$

a verification of the homogeneity. Also

$$(11)\qquad \frac{1}{A}\frac{dM}{dA}+\frac{1}{a}\frac{dM}{du}=\Theta\int\frac{d\theta}{PQ}=4F\frac{\Theta}{r_1}=4F\frac{\sin\psi'}{p},$$

$$(12)\qquad \frac{1}{A}\frac{dM}{dA}-\frac{1}{a}\frac{dM}{da}=\Theta\int\frac{a^2-A^2}{MQ^2}\frac{d\theta}{PQ}=\frac{4B(2fF')}{p},$$

$$(13)\qquad \frac{1}{A}\frac{dM}{dA}=\frac{2}{p}[Fc'\operatorname{sn}2fF'+B(2fF')]=\frac{4B(fK')}{p}$$

$$(14)\qquad \frac{1}{a}\frac{dM}{du}=\frac{2}{p}[Fc'\operatorname{sn}2fF'-B(2fF')]=-\frac{4A(1-f)K'}{p},$$

and we see now that the conditions are satisfied that this M of Viriamu JONES is the Stokes function of MINCHIN's pair of end circular plates of density $\gamma=\pm 1/2\pi p$, corresponding to a helical current $\gamma_h=1$, and

$$(15)\qquad 2M=a\frac{dM}{da}+A\frac{dM}{dA}+b\frac{dM}{db}+M,$$

$$(16)\quad pM=-2a^2\cdot A(1-f)K'+2A^2\cdot B(fK')+b(r_1+r_2)(K-H),$$

in agreement with (6) § 55 above.

58. Thus the magnetic components for the helical current γ_h, axial and transversal, are, for magnetic potential Ω,

$$(1)\qquad \frac{d\Omega}{db}=-\frac{\gamma_h}{2\pi A}\frac{dM}{dA}=-4\gamma B(fK')=-\gamma(2\pi f+4K\operatorname{zn}fK'),$$

and, now keeping p constant in (8) § 57,

$$(2)\qquad \begin{aligned}\frac{d\Omega}{dA}&=\frac{\gamma_h}{2\pi A}\frac{dM}{db}=\frac{\gamma}{A}\int_0^{2\pi}\frac{-aA\cos\theta d\theta}{PQ}\\&=4\gamma\frac{r_1+r_2}{2A}(K-H)=4\gamma\operatorname{dn}fK'\frac{K-H}{k}=4\gamma\frac{K-H}{\operatorname{dn}(1-f)K'}.\end{aligned}$$

With $b = \infty$ we must take $t_1 = \infty$, and the integrals in (10) § 54 reduce to

(3) $$pM = A\Omega' = m^2 \int_{t_3}^{t_2} \frac{t_2 - t\cdot t - t_3}{\tau - t} \frac{dt}{\sqrt{(4\cdot t_2 - t\cdot t - t_3)}}$$

$$= m^2 \int \left(t - t_3 + \tau - t_2 \frac{\tau - t_2\cdot \tau - t_3}{\tau - t} \right) \frac{dt}{\sqrt{(4\cdot t_2 - t\cdot t - t_3)}}$$

(4) $$= m^2 \left[\frac{t_2 - t_3}{2} + \tau - t_2 - \sqrt{(\tau - t_2\cdot\tau - t_3)} \right] \tfrac{1}{2}\pi$$

with

(5) $$4aA = m^2(t_2 - t_3), \quad (a \sim A)^2 = m^2(\tau - t_2), \quad (a + A)^2 = m^2(\tau - t_3),$$

so that

(6) $$pM = \pi A^2 (A < a) \qquad \text{or} \qquad \pi a^2 (A > a).$$

Otherwise, with $b = \infty$ in (5) § 54,

(7) $$pM = A\Omega' = \int_0^{2\pi} \frac{a^2 A^2 \sin^2\theta d\theta}{a^2 + 2aA\cos\theta + A^2} = \pi A^2 \text{ or } \pi a^2,$$

employing the substitution of § 26.

Thus for the infinite helix, in the plane of the end AB,

(8) $$\frac{d\Omega}{db} = -\frac{\gamma_h}{p} = -2\pi\gamma \qquad \text{or} \qquad O,$$

and at M or M',

(9) $$\frac{d\Omega}{dA} = 4\gamma \frac{K_1 - H_1}{k_1} \qquad \text{or} \qquad 4\gamma(K_1 - H_1),$$

(10) $$k_1 = \frac{OM}{OA} \qquad \text{or} \qquad \frac{OA}{OM'}.$$

Thence by subtraction, for a helix extending from b to infinity,

(11) $$\frac{d\Omega}{db} = 4\gamma A(fK'),$$

(12) $$\frac{d\Omega}{dA} = 4\gamma \left[\frac{K - H}{\operatorname{dn}(1 - f)K'} - \frac{K_1 - H_1}{k_1} \right],$$

with $2f \lesseqgtr 1$, according as $A \lesseqgtr a$.

Potential energy of two coaxial helices.

59. There remains still the evaluation of the potential energy of two coaxial helical currents or of their equivalent cylindrical current sheets, given by V. JONES in § 13, p. 202, Proceedings of the Royal Society, 1897, and this is the same as the mutual potential energy of the two pairs of equivalent end plates. Viriamu JONES shows that this energy depends on the integral

$$(1) \qquad F(z) = \frac{Aa}{p}\int_0^{2\pi} \cos\theta d\theta f(z),$$

$$(2) \quad f(z) = z\log\left[\sqrt{\left(1+\frac{z^2}{a^2}\right)}+\frac{z}{a}\right] - \sqrt{(a^2+z^2)}, \quad f'(z) = \left[\log\sqrt{\left(1+\frac{z^2}{a^2}\right)}+\frac{z}{a}\right],$$

$$(3) \qquad F(z) = Mz + \int_0^{2\pi} \frac{-Aa\cos\theta}{p} d\theta\sqrt{(a^2+z^2)},$$

and the second term in $F(z)$,

$$(4) \qquad \begin{aligned} H(z) &= \frac{m^3}{p}\int_{t_3}^{t_2} (2t - t_2 - t_3)(t_1 - t)\frac{dt}{\sqrt{T}} \\ &= \frac{m^3}{p}\int \left((t_2 - t\cdot t - t_3) - \sqrt{T}\frac{d}{dt}\sqrt{T}\right)\frac{dt}{\sqrt{T}}, \end{aligned}$$

the same integral as for G in § 32.

But it is the force between the two helices which alone is required, and this is given by $dF(z)/dz$, or by the change in $F(z)$ due to a small relative axial displacement of the helices, and as this is equivalent to the removal of a circular element from one end to the other of a helix, this force depends only on the difference of the values of M between one helix and the two circular ends of the other helix; and this is calculated by the preceding analysis, which gives the value of M as a function, say $M(b)$, of b the height of one helix say of radius A, when the ring of radius a is in the plane of one base of the helix.

When the circle is at a distance z from the mean section of a helix of height h

$$(5) \qquad M = M(z+\tfrac{1}{2}h) - M(z-\tfrac{1}{2}h);$$

and since $M(-z) = -M(z)$, this makes $M = 2M(\tfrac{1}{2}h)$, in the mean section.

The hydrodynamical analogy is complete, M being the Stokes function, and $V(z+\tfrac{1}{2}h) - V(z-\tfrac{1}{2}h)$ the velocity function, $V(z)$ denoting the potential of a circular plate, due to a uniform distribution of source or sink over the plate, thereby producing a streaming motion circulating through the helix, a line of flow having $M =$ constant: thereby we gain a physical conception of the lines of magnetic force inside and outside a solenoid.

MAXWELL's figure XVIII can be utilised by the method of superposition for drawing these lines when the ring is made to expand into a cylinder by progressive axial displacement.

Exploration of the electromagnetic field.

60. The exploration of the electromagnetic field of the ring current is given by MAXWELL, E. and M. § 702, and in fig. XVIII by drawing the twelve surfaces $\Omega = 2\pi f$, taking

$$(1) \qquad f = (0, 1, 2, 3, \cdots, 11, 12)\tfrac{1}{12};$$

equivalent to direction at the ring on a clock face at B, for every hour and half hour from XII to VI, and again from VI to XII.

Any surface $\Omega = 2\pi f$ will cross the axis Oz at a point H such that

$$(2) \qquad \sin\psi = \sin OBH = \frac{OH}{HB} = 1 - \frac{2\pi f}{2\pi} = 1 - f,$$

so that H is determined geometrically by drawing a circle center A and radius $(1-f)AB$, cutting the circle on diameter AB in H', and joining BH' cutting Oz in H.

Thence at O, where $f = 1$,

$$\frac{d\Omega}{db} = \frac{2\pi(1-f)}{a(1-f)} = \frac{2\pi}{a},$$

the galvanometer constant for the ring.

Practically we need only draw the surfaces from $f = 0$ to $\frac{1}{2}$, or between XII and III o'clock, as the remainder from III to VI can be inferred by the theorems in (25) that $OM \cdot OM' = OB^2$ for points P, P' on a line through B.

In another method of exploration we can utilize the analytical results given in Philosophical Transactions, 1904, and determine the value of Ω, V, and M in §§ 34, 42, 43, 44, 54, when $2f$ is one of the simplest rational fractions, such as

$$(3) \qquad 2f = 1, \frac{2}{3}, \frac{1}{2}, \frac{1}{3}, \frac{1, 3}{4}, \frac{1}{5}, \frac{1}{6}, \cdots.$$

In this method it is simple to work with the elliptic functions to modulus c' and argument $2fF'$ and to use the quadric transformation to obtain the result to argument fK', if required.

A curve along which $2f$ is a rational fraction is an algebraical curve, passing through A at an angle $f\pi$ with Ox, and having Oz as an asymptote.

We proceed to trace a few of the simplest, and for this purpose it is useful to employ a stereographic chart with poles at A and B, and center of projection the antipodes of O, in which case, as in § 25,

$$(4) \qquad k = \tan\tfrac{1}{2}\lambda = \operatorname{th}\alpha, \qquad c' = \tan(45° - \tfrac{1}{2}\lambda) = e^{-2\alpha},$$

λ denoting the latitude, the angle FOL on fig. A.

Exploring the values of Ω, V, M at the principal divisions of the clock face:

XII o'clock, $f = 0$, $b = 0$, $\psi = 0$, $x = ak = a\dfrac{1 - c'}{1 + c'}$,

$$\Omega = 2\pi, \qquad V = 16a\frac{E + c'F}{1 + c'}, \qquad M = 0.$$

VI o'clock, $f = 1$, $b = 0$, $\psi = \pi$, $x = \dfrac{a}{k} = a\dfrac{1 + c'}{1 - c'}$,

$$\Omega = 0, \qquad V = 16a\frac{E - c'F}{1 - c'}, \qquad M = 0.$$

III o'clock, $2f=1$, $\psi=\frac{1}{2}\pi$, $x=a$, $z=2a\dfrac{c'}{c}$, $\sin\psi'=c'$, $\psi=2\,\mathrm{am}\,\frac{1}{2}K'-\frac{1}{2}\pi$,

$$\Omega = \pi - 2Fc', \quad V = 4a\frac{H}{c} - \pi z = z\left(\frac{2H}{c'} - \pi\right), \quad \frac{M}{\Theta} = 2a\frac{F-E}{c} = z\frac{F-E}{c'},$$

on reference to (9) § 55.

II o'clock, $2f = \frac{2}{3}$; and (see Philosophical Transactions, 1904, page 261) in region B, $3 > p > 1$,

$$c^2 = \frac{(p-1)^3(p+3)}{16p}, \qquad c'^2 = \frac{(p+1)^3(-p+3)}{16p},$$

$$\mathrm{zn}\,\tfrac{2}{3}F' = \frac{3-p}{6}\sqrt{p}, \qquad \mathrm{zn}\,\tfrac{2}{3}F'' = \frac{3-p}{6}\sqrt{p},$$

$$\cos\psi = \mathrm{cn}\,\tfrac{2}{3}F' = \frac{p-1}{p+1}, \qquad \cos\psi' = \mathrm{dn}\,\tfrac{2}{3}F' = \frac{p-1}{2}, \qquad \sec\psi - \sec\psi' = 1,$$

along the curve, analogous to

$$\psi + \psi', \text{ or } \tan\psi + \tan\psi', \text{ or } \cos\psi + \cos\psi', \cdots \text{ a constant},$$

a circle, or parabola, or magnetic line of force, $\cdots$ in biangular coördinates ψ, ψ'.

I o'clock: $2f = \frac{1}{3}$, $\sin\Psi = \mathrm{sn}\,\frac{1}{3}F' = \dfrac{2}{p+1}$,

$$\sin\Psi' = c'\,\mathrm{sn}\,\tfrac{1}{3}F' = \sqrt{\frac{p+1\cdot -p+3}{4p}}, \qquad \mathrm{zn}\,\tfrac{1}{3}F' = \frac{9-p^2}{12\sqrt{p}}.$$

I. 30, $2f = \frac{1}{2}$, $\psi = \mathrm{am}\,\frac{1}{2}F'$, $\sin\psi = \sqrt{\dfrac{1}{1+c}}$, $\sin\psi' = \sqrt{(1-c)}$,

$$\tan^2\psi - \tan^2\psi' = 1, \qquad \frac{z^2}{(a-x)^2} - \frac{z^2}{(a+x)^2} = 1,$$

$$\mathrm{zn}\,\tfrac{1}{2}F' = \tfrac{1}{2}(1-c), \qquad \Omega = \tfrac{3}{2}\pi - F(1-c) - 2F\sqrt{(1-c)}.$$

XII. 45, or II. 15, $2f = \frac14$ or $\frac34$ (Philosophical Transactions, page 478),

$$1 > p > \sqrt{2} - 1, \qquad \sqrt{c} = \tfrac12\left(\frac{1}{p} - p\right),$$

$$\cos^2\psi' = \mathrm{dn}^2\tfrac14 F' = \frac{(1+p)^3(1-p)}{4p}, \qquad \sin^2\psi' = \frac{(1+p^2)(-1+2p+p^2)}{4p},$$

$$\tan^2\psi = \frac{4p^3}{(1+p)(1-p)^3}, \qquad \mathrm{zn}\,\tfrac14 F' = \frac{(1+2p+3p^2)(-1+2p+p^2)}{16p^2},$$

and a change of p into $-1/p$ will change $2f$ from $\frac14$ to $\frac34$.

XII. 36, I. 12, I. 48, II. 24; $2f = \frac15, \frac25, \frac35, \frac45$, and the results in Philosophical Transactions, page 264, to be consulted.

XII. 30, II. 30, $2f = \frac16, \frac56$; consult page 283.

61. In an apparatus constructed by Professor AYRTON and Viriamu JONES,

$$2A = 8, \qquad 2a = 13 \text{ inches};$$

and then for the height b we shall find in correspondence with $2f$,

$2f$	0,	$\frac14$,	$\frac13$,	$\frac12$,	$\frac23$,	$\frac34$,	1
b	0,	1.36,	1.47,	2.57,	4.59,	9.79,	∞.

For if

(1) $$2f = \tfrac12, \qquad b = \frac{a^2 - A^2}{2\sqrt{(aA)}} = 2.574,$$

(2) $$2f = \tfrac23, \qquad \left(\frac{a-A}{a+A}\right)^2 = c'^2\,\mathrm{sn}^2(1-2f)F' = \frac{(p+1)(-p+3)}{4p},$$

$$p = 2.8322, \qquad \frac{b}{a-A} = \mathrm{tn}\,\tfrac23 F' = \frac{2\sqrt{p}}{p-1}, \qquad b = 4.593,$$

(3) $$2f = \tfrac13, \qquad \frac{2\sqrt{aA}}{a+A} = \mathrm{dn}\,\tfrac23 F' = \frac{p-1}{2}, \qquad p = 2.9428,$$

$$\frac{b}{a-A} = \mathrm{tn}\,\tfrac13 F' = \frac{2}{\sqrt{(p-1\cdot p+3)}}, \qquad b = 1.471,$$

(4) $$2f = \tfrac34, \qquad \mathrm{dn}^2\,\tfrac14 F' = \frac{4aA}{(a+A)^2} = \frac{416}{441} = \frac{(1+p)^3(1-p)}{4p},$$

a quartic for p having a root $p = 0.43$ and making

$$c = \tfrac14\left(\frac{1}{p} - p\right)^2 = 0.719, \qquad c' = 0.695,$$

$$\cos^2\psi' = \mathrm{dn}^2\,\tfrac34 F' = \tfrac{441}{416}c^2 = 0.547 \qquad (\psi' = 43^\circ),$$

$$b = (a+A)\tan\Psi' = 9.79.$$

Similarly for $2f = \frac{1}{4}$, $b = 1.364$.

62. To determine the proportions of the apparatus when the modular angle γ is raised to 75°, and we take $2f = \frac{1}{3}$, as in fig. A, we put

$$(1) \qquad c^2 = \frac{(p-1)^3(p+3)}{16p} = \sin^2 75^\circ = \frac{2+\sqrt{3}}{4},$$

of which one root is $p = -\sqrt{3}$, but this root leads to

$$(2) \qquad \operatorname{sn} \tfrac{1}{3}F = \sqrt{3} - 1, \qquad \operatorname{dn} \tfrac{1}{3}F = \tfrac{1}{2}\sqrt{2}.$$

But with the other real root,

$$(3) \qquad p = \frac{(\sqrt[3]{2}+1)^2}{\sqrt{3}} = \left(\frac{\sqrt[3]{2}+1}{\sqrt[3]{2}-1}\right)^{\frac{1}{2}} = 2.948661,$$

$$(4) \qquad \csc\psi = \operatorname{sn}\tfrac{1}{3}F' = \tfrac{1}{2}(p+1) = 1.97433 \qquad (\psi = 30^\circ 26'),$$

$$(5) \qquad \sin\psi' = c'\sin\psi = \sqrt{\frac{p+1 . -p+3}{4p}} = 0.1319 \qquad (\psi' = 7^\circ 34'),$$

$$(6) \qquad \sin\psi_1 = \operatorname{sn}\tfrac{2}{3}F' = \frac{2\sqrt{p}}{p+1} \qquad (\psi_1 = 60^\circ 26'),$$

$$(7) \qquad \cos\psi_1' = \operatorname{dn}\tfrac{2}{3}F' = \tfrac{1}{2}(p-1) = 0.97433 \qquad (\psi_1' = 13^\circ 13'),$$

$$(8) \qquad r_1 = (a+A)\sec\psi' = (a+A)\operatorname{dn}\tfrac{1}{3}F' = (a+A)\sqrt{\frac{2p}{p-1\cdot p+3}},$$

$$(9) \qquad \operatorname{zn}(1-2f)F' = \operatorname{zn}\tfrac{2}{3}F' = \tfrac{1}{6}(3p^{\frac{1}{2}} - p^{\frac{3}{2}}),$$

$$(10) \qquad \frac{M}{r_1\Theta} = F - E - \frac{p-1\cdot p+3}{4\sqrt{p}}\left(\frac{\pi}{6} - F\frac{3p^{\frac{1}{2}} - p^{\frac{3}{2}}}{6}\right),$$

$$(11) \qquad F = 2.768063145, \qquad E = 1.076405113, \qquad F - E = 1.691658032.$$

These angles are contrasted with those of the apparatus of Viriamu JONES in the table following.

$2f$	$\frac{1}{3} = 0.33$	0.35
$\sin^{-1} c = \gamma$	75°	74° 1′
$\sin^{-1} k = \beta$	36° 4′	34° 36′
ψ	30° 26′	32° 3′
ψ'	7° 34′	8° 24′
ψ_1	60° 26′	58° 58′
ψ_1'	13° 3′	13° 36′
A/a	0.63196	0.61921

63. If the windings of the helix are not complete, or if the current is considered through a fraction of the circular ring, the theta functions are required or the pseudo-elliptic algebraical III. E. I. can be employed for a rational value of the fraction f, the simplest cases of which will be found worked out in the Philosophical Transactions, 1904.

Lord RAYLEIGH has shown, in a Report to the British Association (1899), that all error in extreme cases of this assumption as between two coaxial helices is eliminated by taking an average for different symmetrical positions.

One object of the present memoir is to see if by a slight change in one of the dimensions of a helix, say the height b, designed for the experimental determination of the ampere, it would be possible to make $2f$ one of the rational fractions given above, and so simplify and check the numerical work.

Or we may construct the apparatus so as to make the angle γ in fig. A or β in fig. B an integral number of degrees, so that the value of F and E or K and H can be taken out of LEGENDRE's Table I and IX without use of proportional parts, and utilise all the decimal figures.

Potential and Stokes function of a solid coil.

64. To solve Coleridge FARR's problem for a solid cylindrical coil, we have to determine the potential W and Stokes function P of a series of coaxial helical currents or equivalent cylindrical current sheets filling up a solid cylinder, integrating with respect to the radius the expression for a cylindrical sheet.

It is simpler to determine W for a coil stretching in fig. A from the plane AB to infinity on the right, as this is equivalent to integrating over a single circular plate AB of superficial density $\gamma = \gamma_h/2\pi p$ (qualified as shown in (8) by the factor $a - r$, r referring to any internal radius OQ'), the influence of the other end plate at infinity being insensible.

In § 48 we have found for the potential V of the plate AB of unit density

$$(1) \qquad \frac{dV}{d\theta} = PQ - PO - A\cos\theta \cdot 2\,\mathrm{th}^{-1}\frac{a}{PQ+PO},$$

the equivalent of $d^2W/da\,d\theta$, and integrating by parts with respect to a,

$$(2) \qquad \frac{dW}{d\theta} = \int_0^a (PQ - PO)\,da - A\cos\theta\int 2\,\mathrm{th}^{-1}\frac{a}{PQ+PO}\,da$$

$$= (\tfrac{1}{2}a + \tfrac{3}{2}A\cos\theta)PQ - (a + \tfrac{3}{2}A\cos\theta)PO$$

$$- 2\,\mathrm{th}^{-1}\frac{a}{PQ+PO}(\tfrac{1}{4}A^2 - \tfrac{1}{2}b^2 + aA\cos\theta + \tfrac{3}{4}A^2\cos 2\theta).$$

Integrating again with respect to θ from 0 to 2π, the term

$$(3)\qquad I = \int 2\mathrm{th}^{-1}\frac{a}{PQ + PO}\,d\theta$$

is not an elliptic integral, and so considered intractable, like the integral (2) in § 53, to be denoted by I_1; but

$$\int aA\cos\theta\cdot 2\mathrm{th}^{-1}\frac{a}{PQ+PO}\,d\theta = a\int\left(1-\frac{b^2}{PZ^2}\right)(aA\cos\theta + A^2 + b^2)\frac{d\theta}{PQ}$$

$$(4)\qquad -a\sqrt{(A^2+b^2)}\int\left(1-\frac{b^2}{PZ^2}\right)d\theta$$

$$= \int(aA\cos\theta + A^2 + b^2)\,a\frac{d\theta}{PQ} + ab[(2\pi - \Omega(PZ)] - 2\pi a\sqrt{(A^2+b^2)}.$$

$$\int A^2\cos 2\theta\cdot 2\mathrm{th}^{-1}\frac{a}{PQ+PO}\,d\theta = -aA^2\int\frac{\sin^2\theta\,d\theta}{PQ}$$

$$(5)\qquad +(A^2+b^2)\int(a + A\cos\theta)\frac{d\theta}{PQ} - b\sqrt{(A^2+b^2)}\,\Omega_1(PZ),$$

so that W is composed of

$$(6)\quad (\tfrac{1}{2}b^2 - \tfrac{1}{4}A^2)I, \qquad -ab[2\pi - \Omega(PZ)], \qquad \tfrac{3}{4}b\sqrt{(A^2+b)}\,\Omega_1(PZ),$$

and I. and II. E. I.'s, amounting to employing the theorems of § 33,

$$\frac{aF}{2r_1}(a^2 - A^2 - 13b^2) + \frac{3ar_1E}{2},$$

$$2W = (b^2 - \tfrac{1}{2}A^2)I - 2ab[2\pi - \Omega(PZ)] + \tfrac{3}{2}b\sqrt{(A^2+b^2)}\,\Omega_1(PZ)$$

$$(7)\qquad + \frac{aF}{r_1}(a^2 - A^2 - 13b^2) + 3ar_1E.$$

The potential W may also be considered due to the circular plate AB, supposing the density at any interior radius $OQ' = r$ is $a - r$; and then, expressed as a double integral

$$(8)\qquad W = \iint\frac{(a-r)r\,dr\,d\theta}{PQ'},$$

$$(9)\qquad \frac{dW}{da} = \iint\frac{r\,dr\,d\theta}{PQ'} = \text{potential of the end plate of uniform density}$$

$$= \int(a^2 + aA\cos\theta)\frac{d\theta}{PQ} - b\,[2\pi - \Omega(PZ)],$$

$$(10)\qquad \frac{dW}{dA} = \iint\frac{-(a-r)(r\cos\theta + A)r\,dr\,d\theta}{PQ'^3},$$

$$(11)\qquad \frac{dW}{db}=\iint\frac{-(a-r)rdr\cdot bd\theta}{PQ'^3};$$

and

$$(12)\qquad a\frac{dW}{da}+A\frac{dW}{dA}+b\frac{dW}{db}=2W,$$

the homogeneity condition of W, of two dimensions in length.

65. Other double integrals required in the calculation are

$$(1)\qquad 2\pi-\Omega(PZ)=\iint\frac{rdr\cdot bd\theta}{PQ'^3},$$

the apparent area or normal attraction of the disc AB;

$$(2)\qquad \Omega_1(PZ)=\iint\frac{\sqrt{(A^2+b^2)}\,dr\cdot bd\theta}{PQ'^3},$$

$$(3)\qquad I=\iint\frac{dr\,d\theta}{PQ'},$$

the intractable integral (3) § 64 which is the potential of the disc AB, with density r^{-1};

$$\iint\frac{r\cos\theta\,dr\,d\theta}{PQ'^3}=\iint\frac{(r+A\cos\theta)\cos\theta-A\cos^2\theta}{PQ'^3}\,drd\theta$$

$$(4)\qquad =\int\left(\frac{1}{PO}-\frac{1}{PQ}\right)\cos\theta d\theta-\int\frac{A\cos^2\theta}{PZ^2}\left(\frac{a+A\cos\theta}{PQ}-\frac{A\cos\theta}{PO}\right)d\theta$$

$$=\frac{a}{A}\int\frac{d\theta}{PQ}-\frac{\sqrt{(A^2+b^2)}}{Ab}\Omega_1(PZ)+\sqrt{(A^2+b^2)}\int\frac{\cos\theta d\theta}{PZ^2},$$

from (17) § 47, and the third integral vanishes;

$$(5)\qquad \iint\frac{r^2dr\,d\theta}{PQ'^3}=I-2a\int\frac{d\theta}{PQ}+\frac{\sqrt{(A^2+b^2)}}{b}\Omega_1(PZ),$$

$$(6)\qquad \frac{1}{b}\frac{dW}{db}=I-2a\int\frac{d\theta}{PQ}+\frac{\sqrt{(A^2+b^2)}}{b}\Omega_1(PZ)-\frac{a}{b}[2\pi-\Omega(PZ)],$$

$$(7)\qquad \begin{aligned}A\frac{dW}{dA}&=2W-a\frac{dW}{du}-b\frac{dW}{db}\\&=-\tfrac{1}{2}A^2I+\tfrac{1}{2}b\sqrt{(A^2+b^2)}\,\Omega_1(PZ)-\frac{aF}{r_1}(a^2-A^2+3b^2)+ar_1E,\end{aligned}$$

$$(8)\qquad \frac{dP}{db}=-2\pi A\frac{dW}{dA},\qquad \frac{dP}{dA}=2\pi A\frac{dW}{db}.$$

66. The Stokes function P can be found by an integration of Viriamu Jones's M, but an infinite constant comes in when the coil stretches from b to ∞; so returning to a coil extending from 0 to b, or from AB to the parallel plane through P,

$$(1)\qquad \frac{d^2P}{da\,d\theta}=\frac{dpM}{d\theta}=\frac{a^2A^2\sin^2\theta}{MQ^2}\frac{b}{PQ},$$

and since

$$(2)\qquad a^2=MQ^2-2A\cos\theta(a+A\cos\theta)+A^2\cos 2\theta,$$

$$(3)\qquad \begin{aligned}\frac{dP}{d\theta}&=A^2\sin^2\theta\int\frac{bda}{PQ}-2A^2\sin^2\theta\cos\theta\int\frac{a+A\cos\theta}{MQ^2}\frac{bda}{PQ}\\&\qquad+A^2\sin^2\theta\cos 2\theta\int\frac{1}{MQ^2}\frac{bda}{PQ}\\&=A^2b(1-\cos 2\theta)\,\mathrm{th}^{-1}\frac{a}{PQ+PO}+2A^3\sin^2\theta\cos\theta\left(\mathrm{th}^{-1}\frac{b}{PQ}-\mathrm{th}^{-1}\frac{b}{PO}\right)\\&\qquad-A^3\sin\theta\cos 2\theta\left[I_2-\tan^{-1}\left(\frac{PO}{b}\tan\theta\right)\right],\end{aligned}$$

where I_2 is given in (7) § 47, and the integration of this term by parts with respect to θ will depend on $\Omega(PZ)$ and $\Omega_1(PZ)$; and P is thus composed of $\frac{1}{2}A^2bI$ and I., II., and III. E. I.'s, which can be evaluated in the preceding manner.

The intractable integral

$$(4)\qquad I=\int 2\,\mathrm{th}^{-1}\frac{a}{PQ+PO}d\theta=\int\mathrm{th}^{-1}\frac{a+A\cos\theta}{PQ}d\theta-\int\mathrm{th}^{-1}\frac{A\cos\theta}{PO}d\theta,$$

of which the second integral vanishes.

Putting $(a+A\cos\theta)/PQ=\sin\phi'$, so that ϕ' is the angle between PO and PZ, and drawing the curve on the Mercator chart connecting latitude ϕ' and longitude θ, the area will represent I, as before in § 53 for I_1.

For a flat coil in the plane AB,

$$(5)\qquad P=\iint\frac{-2\pi Ar\cos\theta\,drd\theta}{PQ'},$$

$$(6)\qquad \frac{dP}{d\theta}=-2\pi A\cos\theta(PQ-PO)+\pi A^2(1+\cos 2\theta)\,\mathrm{th}^{-1}\frac{a}{PQ+PO},$$

so that

$$(7)\qquad P=\tfrac{1}{2}\pi A^2I\text{ and I., II., and III. E. I.'s.}$$

Legendre's discussion of the oblique cone.

67. The method of discussion by LEGENDRE of the surface of the oblique cone in his *Fonctions elliptiques* I, p. 329, may be resumed here, because his Φ, the angle of the sector of the developed surface, and Ω the conical angle of opening of the reciprocal cone are connected by the relation

$$(1) \qquad \Phi + \Omega = 2\pi,$$

so that one calculation implies the other.

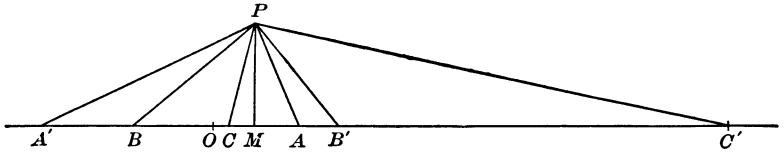

FIG. *C*.

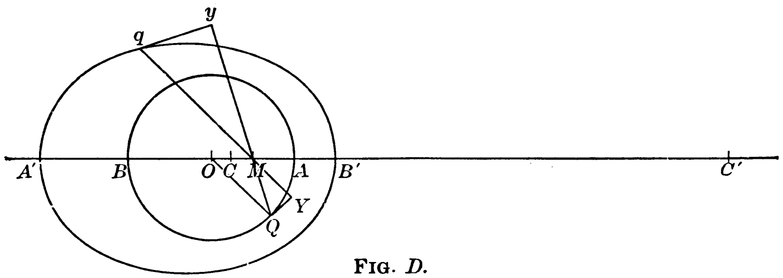

FIG. *D*.

Adopting LEGENDRE's notation for his oblique cone on a circular base, figures *C* and *D*, and considering it as the reciprocal of the cone discussed previously in figs. *A* and *B*, then PM the perpendicular from the vertex P on the plane of the circular base AB making an angle λ with PC the bisector of the angle APB, and the edges PA', PB' of the reciprocal cone making an angle λ' with PC, and putting (LEGENDRE)

$$(2) \quad AO = OB = r, \qquad OM = f, \qquad MP = h, \qquad PA = \alpha, \qquad PB = \alpha',$$

$$(3) \qquad \sin PAC = \sin(\lambda' + \lambda) = \frac{h}{\alpha}, \qquad \cos(\lambda' + \lambda) = \frac{r - f}{\alpha},$$

$$(4) \qquad \sin PBC = \sin(\lambda' - \lambda) = \frac{h}{\alpha}, \qquad \cos(\lambda' - \lambda) = \frac{r + f}{\alpha'},$$

$$(5) \qquad \cos 2\lambda = \frac{r^2 - f^2 + h^2}{\alpha\alpha'}, \qquad \sin 2\lambda = \frac{2fh}{\alpha\alpha'},$$

$$(6) \qquad \cos 2\lambda' = \frac{r^2 - f^2 - h^2}{\alpha\alpha'}, \qquad \sin 2\lambda' = \frac{2rh}{\alpha\alpha'},$$

$$(7) \qquad \cos\lambda \sin\lambda' = \frac{h(\alpha' + \alpha)}{2\alpha\alpha'}, \qquad \sin\lambda \cos\lambda' = \frac{h(\alpha' - \alpha)}{2\alpha\alpha'},$$

$$(8) \qquad \frac{\tan\lambda}{\tan\lambda'} = \frac{\alpha' - \alpha}{\alpha' + \alpha} = \cos\theta,$$

in LEGENDRE's notation.

If Pq is the generating line of the reciprocal cone perpendicular to the plane PQY, making an angle β with PM, and c denotes cos AOQ,

$$(9)\qquad \cot\beta = \frac{MY}{PM} = \frac{r - fc}{h} = \frac{\sin 2\lambda' - c \operatorname{am} 2\lambda}{\cos 2\lambda - \cos\lambda'},$$

$$(10)\qquad \begin{aligned} \cos\beta\cos 2\lambda' + \sin\beta\sin 2\lambda' &= \cos(2\lambda' - \beta) \\ &= \cos\beta\cos 2\lambda + c\sin\beta\sin 2\lambda' = \cos\beta', \end{aligned}$$

if β' denotes the angle between PQ' and PM' the perpendicular on the other circular section of the cone in fig. D; so that

$$(11)\qquad \beta + \beta' = 2\lambda',$$

or the sum of the angles which PQ' makes with PM, PM' is constant, and these are the focal lines of the reciprocal cone in fig. A; hence the fundamental property of a sphero-conic.

68. LEGENDRE denotes the angle of the sector APQ when developed by Φ, and the angle AOQ by ω, so that

$$(1)\qquad PQ^2 d\Phi = PY \cdot r d\omega,$$

$$(2)\qquad \Phi = \int \frac{PY^2}{PQ^2} \cdot \frac{r d\omega}{PY},$$

and, writing c for $\cos\omega$,

$$(3)\qquad PQ^2 = r^2 - 2rfc + f^2 + h^2, \qquad PY^2 = (r - fc)^2 + h^2.$$

The angle QPY is the ψ of LEGENDRE, and to reduce Φ to our form, put

$$(4)\qquad m^2(\sigma - s) = \frac{PQ^2}{PY^2} = \sec^2\psi = \frac{r^2 - 2rfc + f^2 + h^2}{(r - fc)^2 + h^2},$$

$$(5)\qquad m^2(s_2 - s) = \frac{QY^2}{PY^2} = \tan^2\psi = \frac{f^2(1 - c^2)}{(r - fc)^2 + h^2},$$

$$(6)\qquad m^2(\sigma - s_2) = 1.$$

Then, as before in § 29,

$$(7)\qquad m^2(s_1 - s) = \frac{\alpha\alpha'}{h^2}\cos^2\lambda\left(\frac{(\alpha' + \alpha) - (\alpha' - \alpha)c}{2PY}\right)^2,$$

$$(8)\qquad m^2(s - s_3) = \frac{\alpha\alpha'}{h^2}\sin^2\lambda\left(\frac{(\alpha' + \alpha)c - (\alpha' - \alpha)}{2PY}\right)^2.$$

Putting $s = s_2$, $c = \pm 1$,

$$(9)\qquad \begin{aligned} &m^2(s_1 - s_2) = \frac{\alpha\alpha'}{h^2}\cos^2\lambda, \qquad m^2(s_2 - s_3) = \frac{\alpha\alpha'}{h^2}\sin^2\lambda, \\ &m^2(s_1 - s_3) = \frac{\alpha\alpha'}{h^2}, \qquad m^4(s_1 - s_2)(s_2 - s_3) = \frac{f^2}{h^2}, \end{aligned}$$

$$(10)\qquad \sin^2\lambda = \frac{s_2 - s_3}{s_1 - s_3} = k^2,$$

so that λ is the modular angle,

$$(11)\qquad m^2(s_1 - \sigma) = \frac{\alpha\alpha'}{h^2}\cos^2\lambda - 1 = \frac{\alpha\alpha'}{h^2}\cos^2\lambda',$$

$$(12)\qquad m^2(\sigma - s_3) = \frac{\alpha\alpha'}{h^2}\sin^2\lambda + 1 = \frac{\alpha\alpha'}{h^2}\sin^2\lambda,$$

$$(13)\qquad \tfrac{1}{2}m^3\sqrt{(-\Sigma)} = \frac{r}{h}.$$

Again from (11) § 67,

$$\tan(\beta - \lambda') = \tan(\lambda' - \beta') = \tan\tfrac{1}{2}(\beta - \beta')$$

$$(14)\qquad = \tan\lambda\frac{c\cos\lambda\sin\lambda' - \sin\lambda\cos\lambda'}{\cos\lambda\sin\lambda' - c\sin\lambda\cos\lambda'} = \tan\lambda\frac{(\alpha' + \alpha)c - (\alpha' - \alpha)}{\alpha' + \alpha - (\alpha' - \alpha)c} = \sqrt{\frac{s - s_3}{s_1 - s}},$$

so that as before in § 35,

$$(15)\qquad d\beta = \frac{fhdc}{(r - fc)^2 + h^2} = d\tan^{-1}\sqrt{\frac{s - s_3}{s_1 - s}} = \frac{ds}{\sqrt{(4\cdot s_1 - s\cdot s - s_3)}},$$

$$(16)\qquad \frac{rd\omega}{PY} = \frac{-rdc}{PY\sqrt{(1 - c^2)}} = \frac{-frdc}{PY^2\cdot m\sqrt{(s - s_2)}} = -\frac{r}{h}\frac{ds}{m\sqrt{S}}.$$

$$(17)\qquad \Phi = \int_{s_3}^{s_2}\frac{\sigma - s_2}{\sigma - s}\frac{r}{h}\frac{ds}{\sqrt{S}} = \int\frac{\frac{1}{2}\sqrt{(-\Sigma)}}{\sigma - s}\frac{ds}{\sqrt{s}}.$$

We notice now that in LEGENDRE's notation, with

$$(18)\qquad \cos\theta = \frac{\alpha' - \alpha}{\alpha' + \alpha}, \qquad \text{and} \qquad \tan\tfrac{1}{2}\omega = \tan\tfrac{1}{2}\theta\tan\tfrac{1}{2}\phi,$$

equivalent geometrically to a change from the excentric anomaly ω to the true anomaly ϕ in an orbit of excentricity $\cos\theta$,

$$(19)\qquad \frac{(\alpha' + \alpha)c - (\alpha' - \alpha)}{\alpha' + \alpha - (\alpha' - \alpha)c} = \frac{\cos\omega - \cos\theta}{1 - \cos\omega\cos\theta} = \cos\phi,$$

$$(20)\qquad \sqrt{\frac{s - s_3}{s_1 - s}} = \tan\lambda\cos\phi, \qquad \sqrt{\frac{s - s_3}{s_2 - s_3}} = \frac{\cos\phi}{\Delta\phi}$$

and the elliptic functions here are a quarter-period out of phase with the preceding in § 26, and

$$(21)\qquad \phi = \operatorname{am}(1 - h)K.$$

Also

$$(22)\qquad \frac{\operatorname{cn}^2 fK'}{\operatorname{dn}^2 fK'} = \frac{\sigma - s_2}{\sigma - s_3}\frac{s_1 - s_3}{s_1 - s_2} = \frac{h^2}{\alpha\alpha'}\frac{1}{\sin^2\lambda'\cos^2\lambda} = \frac{4\alpha\alpha'}{(\alpha' + \alpha)^2} = \sin^2\theta,$$

so that

$$(23)\qquad \theta = \operatorname{am}(1 - f)K'.$$

Stereographic projection of confocal sphero-conics.

69. Describe the unit sphere with center P, cutting the focal lines PM, PM' in A, B and PQ' in C, so that in the spherical triangle ABC (fig. E)

$$AB = c = 2\lambda, \qquad BC + CA = a + b = \beta + \beta' = 2\lambda', \tag{1}$$

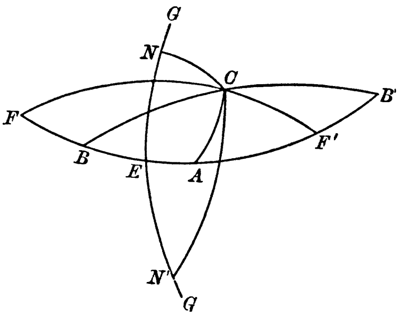

FIG. *E*.

and the locus of C is a sphero-conic for constant $a + b$ or λ' or f, the orthogonal confocal sphero-conics having $a - b$ or h constant.

Projecting these confocal sphero-conics stereographically on the tangent plane at F, the middle point of $A'B$, where the spherical triangle $A'BC$ is colunar with ABC, and putting $CF = \theta$, $CFB = \psi$, we can put, according to BURNSIDE, Messenger of Mathematics, vol. 20 (1891), p. 60,

$$\tan\tfrac{1}{2}\theta e^{\psi i} = \sqrt{c'}\,\text{sn}\,(v + ui,\, c') \qquad [c' = \tan\tfrac{1}{4}(\pi - c)], \tag{2}$$

and obtain the orthogonal system of curves given by HOLZMÜLLER in *Isogonale Verwandtschaften.*

Here $FA = 90 + \tfrac{1}{2}c$, $FB = 90 - \tfrac{1}{2}c$,

$$\begin{aligned} \cos a &= \cos\theta \sin\tfrac{1}{2}c + \sin\theta\cos\tfrac{1}{2}c\cos\phi\cos b \\ &= -\cos\theta\sin\tfrac{1}{2}c + \sin\theta\cos\tfrac{1}{2}c\cos\phi. \end{aligned} \tag{3}$$

$$2\cos\theta\sin\tfrac{1}{2}c = \cos a - \cos b, \qquad \cos\theta = \frac{\sin\frac{1}{2}(b-a)\sin\frac{1}{2}(b+a)}{\sin\frac{1}{2}a}. \tag{4}$$

But, writing w, w' for $v \pm ui$

$$\tan\tfrac{1}{2}\theta = c'\,\text{sn}\,w\,\text{sn}\,w', \qquad \cos\theta = \frac{1 - c'\,\text{sn}\,w\,\text{sn}\,w}{1 + c'\,\text{sn}\,w\,\text{sn}\,w'}. \tag{5}$$

$$\cos\theta = \frac{1 - c'\,\text{sn}^2 v}{1 + c'\,\text{sn}^2 v}\cdot\frac{1 + c'\,\text{sn}^2 ui}{1 - c'\,\text{sn}^2 ui}, \tag{6}$$

so that replacing v and u by fF' and hF, and employing the quadric transformation,

$$\cos\theta = \frac{\text{dn}\,fK'}{\text{dn}\,2hKi}, \tag{7}$$

$$\sin\tfrac{1}{2}(b + a) = \text{dn}\,fK', \qquad \cos\tfrac{1}{2}(b + a) = K'\,\text{sn}\,fK', \tag{8}$$

$$\frac{\sin\frac{1}{2}(b - a)}{\sin\frac{1}{2}c} = \frac{1}{\text{dn}\,2Ki} = \frac{\text{cn}\,2hK}{\text{dn}\,2hK} = \text{sn}\,(1 - 2h)K, \tag{9}$$

$$\sin\tfrac{1}{2}(b - a) = K\,\text{sn}\,(1 - 2h)\,K, \tag{10}$$

$$\cos\tfrac{1}{2}(b - a) = \text{dn}\,(1 - 2h)K, \tag{11}$$

$$k = \sin\tfrac{1}{2}c, \qquad k' = \cos\tfrac{1}{2}c. \tag{12}$$

Thus $h = \frac{1}{2}$ makes $a = b$, and the stereographic projection of the great circle GH is a circle, the coalescence of two branches of a quartic curve in HOLZMÜLLER's diagram.

From NAPIER's formula

$$(13) \qquad \frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)} = \frac{\tan\frac{1}{2}(a+b)}{\tan\frac{1}{2}c} = \frac{\operatorname{dn} fK'}{\kappa \operatorname{sn} fK'} = \frac{1}{\operatorname{cn}(1-f)K'}$$

we deduce

$$(14) \qquad \tan\tfrac{1}{2}A \tan\tfrac{1}{2}B = \frac{1-\operatorname{cn}(1-f)K'}{1+\operatorname{cn}(1-f)K'};$$

and from

$$(15) \qquad \frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)} = \frac{\tan\frac{1}{2}(a-b)}{\tan\frac{1}{2}c} = \frac{k' \operatorname{sn}(2h-1)K}{\operatorname{dn}(2h-1)K} = -\operatorname{cn} 2hK$$

we deduce

$$(16) \qquad \tan\tfrac{1}{2}A \div \tan\tfrac{1}{2}B = \frac{1-\operatorname{cn} 2hK}{1+\operatorname{cn} 2hK},$$

$$(17) \qquad \tan^2\tfrac{1}{2}A = \frac{1-\operatorname{cn}(1-f)}{1+\operatorname{cn}(1-f)} \cdot \frac{1-\operatorname{cn} 2h}{1+\operatorname{cn} 2h},$$

$$(18) \quad \operatorname{cn} A = \frac{\operatorname{cn}(1-f)+\operatorname{cn} 2h}{1+\operatorname{cn}(1+f)\operatorname{cn} 2h}, \qquad \sin A = \frac{\operatorname{sn}(1-f)\operatorname{sn} 2h}{1+\operatorname{cn}(1-f)\operatorname{cn} 2h},$$

$$(19) \qquad \tan^2\tfrac{1}{2}B = \frac{1-\operatorname{cn}(1-f)}{1+\operatorname{cn}(1-f)} \cdot \frac{1+\operatorname{cn} 2h}{1-\operatorname{cn} 2h},$$

$$(20) \quad \cos B = \frac{\operatorname{cn}(1-f)-\operatorname{cn} 2h}{1-\operatorname{cn}(1-f)\operatorname{cn} 2h}, \qquad \sin B = \frac{\operatorname{sn}(1-f)\operatorname{sn} 2h}{1-\operatorname{cn}(1-f)\operatorname{cn} 2h},$$

and so on.

70. If N, N' are the poles of the circular sections of the cone, or the foci of the sphero-conics of the reciprocal cone,

$$(1) \qquad \sin EN = \frac{\tan\frac{1}{2}c}{\tan\frac{1}{2}(a+b)} = \cos\theta = \operatorname{cn}(1-f)K',$$

and we find by Spherical Trigonometry

$$(2) \qquad \cos CN,\ CN' = \cos CE \cos EN \pm \sin CE \sin EN \sin CEA$$

and

$$(3) \qquad \cos CE = \frac{\cos\frac{1}{2}(a+b)\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}c} = \operatorname{sn} f \operatorname{dn}(1-2h),$$

$$(4) \qquad \cos EN = \frac{\sqrt{\sin s \sin(s-c)}}{\cos\frac{1}{2}c \sin\frac{1}{2}(a+b)} = \operatorname{sn}(1-f),$$

$$(5) \quad \sin CE \sin CEA = \frac{\sqrt{\{\sin s \sin(s-a)\sin(s-b)\sin(s-c)\}}}{\sin\frac{1}{2}c\cos\frac{1}{2}c} = \operatorname{cn} f \operatorname{cn}(1-2h),$$

so that

$$(6) \quad \cos CN,\ CN' = \frac{\operatorname{sn} f \operatorname{cn} f}{\operatorname{dn} f}\left[\operatorname{dn}(1-2h) \pm k \operatorname{cn}(1-2h)\right] = \operatorname{sn} 2fF' \operatorname{dn}(\tfrac{1}{2} \pm h)F,$$

and

$$(7)\qquad \cos CN \cos CN' = c' \operatorname{sn}^2 2fF', \text{ a constant,}$$

a fundamental property of the sphero-conic.

71. If dA denotes the element of plane area cut out by consecutive quartic curves,

$$(8)\qquad dA = c' \operatorname{cn} w \operatorname{dn} w \operatorname{cn} w' \operatorname{dn} w' \cdot F \cdot dh \cdot F' \cdot df$$

and if dw denotes the corresponding conical angle,

$$(9)\qquad dw = 4 \cos \Psi \tfrac{1}{2}\theta dA = 4 \frac{c' \operatorname{cn} w \operatorname{dn} w \operatorname{cn} w' \operatorname{dn} w'}{(1 + c' \operatorname{sn} w \operatorname{sn} w')^2} FF' df dh,$$

and putting $w, w' = \frac{1}{2}fK' \pm hKi$, this becomes by the quadric transformation,

$$dK = 4 \frac{\dfrac{1-k}{1+k} \dfrac{(\operatorname{dn}^4 w - k^2)(\operatorname{dn}^4 w' - k^2)}{(1-k^2)^2 \operatorname{dn}^2 w \operatorname{dn}^2 w'}}{\left(1 + \dfrac{k'^2 \operatorname{sn} w \operatorname{cn} w \operatorname{sn} w' \operatorname{cn} w'}{\operatorname{dn} w \operatorname{dn} w'}\right)^2} \tfrac{1}{2}(1+k)^2 KK' df dh$$

$$(10)\qquad = \frac{2}{1-k^2} \frac{(\operatorname{dn}^4 w - k^2)(\operatorname{dn}^4 w' - k^2)}{(\operatorname{dn} w \operatorname{dn} w' + k'^2 \operatorname{sn} w \operatorname{cn} w \operatorname{sn} w' \operatorname{cn} w')^2} KK' df dh$$

$$= 2 \frac{\operatorname{dn}^2 (w + w') \operatorname{dn}^2 (w - w') - k^2}{\operatorname{dn}^2 (w - w')} KK' df dh$$

$$= 2 \left(\operatorname{dn}^2 fK' - \frac{k^2}{\operatorname{dn}^2 2hKi}\right) KK' df dh,$$

in which the variables are separated.

Performing the integrations between limits such that h ranges from 0 to 1, and f from f to 1 and doubling for the whole conical angle Ω,

$$(11)\qquad \iint \operatorname{dn}^2 fK' KK' df dh = K[(1-f)H' - \operatorname{zn} fK'],$$

$$\iint \frac{k^2}{\operatorname{dn}^2 2hKi} KK' df dh = \iint [1 - \operatorname{dn}^2(1-2h)K]\, KK' df dh$$

$$(12)\qquad = (1-f)K' \int_0^1 [1 - \operatorname{dn}^2(1-2h)K]\, K dh$$

$$= (1-f)K'(K - H),$$

so that

$$(13)\ \Omega = 4(1-f)(KH' + K'H - KK') - 4K \operatorname{zn} fK' = 2\pi(1-f) - 4K \operatorname{zn} fK',$$

as before, for the complete cone; but (10) shows that the apparent area of any curvilinear quadrilateral, bounded by two pairs of HOLZMÜLLER's curves can be expressed in the same manner by the I. and II. E. I.

Looking back over the calculation we conclude that the Stokes function has the advantage in simplicity over the potential function, except for the case of the circular ring of § 44.

1 STAPLE INN, LONDON, W. C.,
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